# Ph125b Monday 12 February 2007

## Wave mechanics in more than one dimension

At the very beginning of our discussion of particle motion in one (kinetic) dimension, we noted that the relevant quantum states should live in an infinite-dimensional Hilbert space whose orthogonal basis kets correspond to points on the line:

$$|\Psi\rangle \in H_\infty = \text{span}\{|x\rangle\}.$$

Now we need to move on to two and three kinetic dimensions, and the simple mathematical tool for doing so is our friend the tensor product.

The quantum state of a point particle moving in two kinetic dimensions lives in a doubly-infinite Hilbert space,

$$|\Psi\rangle \in H_\infty \otimes H_\infty = \text{span}\{|x\rangle \otimes |y\rangle\}.$$

And in three dimensions,

$$|\Psi\rangle \in H_\infty \otimes H_\infty \otimes H_\infty = \text{span}\{|x\rangle \otimes |y\rangle \otimes |z\rangle\}.$$

Easy! Wave-functions correspondingly become (scalar) functions of a vector argument:

$$2D: \quad \psi(x,y) = (\langle x| \otimes \langle y|)|\Psi\rangle,$$

$$3D: \quad \psi(x,y,z) = (\langle x| \otimes \langle y| \otimes \langle z|)|\Psi\rangle.$$

It is important to note that changes of “coordinate” basis are possible once we have gone to a wave-function representation, such that $\psi(x,y)$ can equivalently be expressed as $\psi(\rho,\varphi)$, or $\psi(x,y,z)$ as $\psi(r,\theta,\varphi)$. Hence we will often simply write $\psi(\vec{r})$, and things like inner products or operator moments can generally be integrated in whichever coordinate system is most convenient.

$$\langle\Psi|\Psi\rangle = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz\, \psi^*(x,y,z)\psi(x,y,z)$$

$$= \int_0^{+\infty} dr \int_0^{\pi} r\sin\theta d\theta \int_0^{2\pi} rd\varphi\, \psi^*(r,\theta,\varphi)\psi(r,\theta,\varphi),$$

$$\langle\Psi|\mathbf{x}|\Psi\rangle = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz\, \psi^*(x,y,z)\, x\, \psi(x,y,z)$$

$$= \int_0^{+\infty} dr \int_0^{\pi} r\sin\theta d\theta \int_0^{2\pi} rd\varphi\, \psi^*(r,\theta,\varphi)\, r\sin\theta\cos\varphi\, \psi(r,\theta,\varphi).$$

Rather than going on about this too long, let’s just jump into an example.

## The 2D harmonic oscillator; angular momentum states

The 2D harmonic oscillator has potential energy

$$V(x,y) = \frac{1}{2}m\omega_0^2(\mathbf{x}^2 + \mathbf{y}^2).$$

In fact the $x$ and $y$ curvatures could be different, but let’s stick with the degenerate case. For 2D motion, the overall Hamiltonian is thus

$$\mathbf{H} = \frac{|\vec{\mathbf{p}}|^2}{2m} + V(x,y)$$
$$= \frac{1}{2m}\left(\mathbf{p}_x^2 + \mathbf{p}_y^2\right) + \frac{1}{2}m\omega_0^2(\mathbf{x}^2 + \mathbf{y}^2).$$

Here

$$\vec{\mathbf{p}} = -i\hbar\nabla,$$
$$\mathbf{p}_x = -i\hbar\frac{\partial}{\partial x},$$
$$\mathbf{p}_y = -i\hbar\frac{\partial}{\partial y}.$$

We clearly see that the overall Hamiltonian separates into terms that act on the $x$ and $y$ spaces only,

$$\mathbf{H} = \left[\frac{\mathbf{p}_x^2}{2m} + \frac{1}{2}m\omega_0^2\mathbf{x}^2\right] + \left[\frac{\mathbf{p}_y^2}{2m} + \frac{1}{2}m\omega_0^2\mathbf{y}^2\right]$$
$$= \mathbf{H}_x + \mathbf{H}_y,$$

where we understand that $\mathbf{p}_x$ means something like $\mathbf{p}_x \otimes \mathbf{1}^y$, etc. Thus, we have every right to expect that the energy eigenstates will be of tensor-product form, and indeed that the $x$ and $y$ components will correspond to familiar eigenstates of the 1D harmonic oscillator!

To make this most transparent, let's define annihilation operators

$$\mathbf{a}_x = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\mathbf{x} + i\frac{\mathbf{p}_x}{m\omega_0}\right),$$
$$\mathbf{a}_y = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\mathbf{y} + i\frac{\mathbf{p}_y}{m\omega_0}\right).$$

(If we had originally defined different curvatures in the $x$ and $y$ directions, we would define these using $\omega_x$ and $\omega_y$.) In terms of these,

$$\mathbf{H}_x = \hbar\omega_0\left(\mathbf{a}_x^\dagger\mathbf{a}_x + \frac{1}{2}\right),$$
$$\mathbf{H}_y = \hbar\omega_0\left(\mathbf{a}_y^\dagger\mathbf{a}_y + \frac{1}{2}\right).$$

We now simply define number states in the $x$ and $y$ spaces,

$$\mathbf{a}_x^\dagger\mathbf{a}_x|n_x\rangle = n_x|n_x\rangle,$$
$$\mathbf{a}_y^\dagger\mathbf{a}_y|n_y\rangle = n_y|n_y\rangle,$$

and the overall energy eigenstates are $|n_x n_y\rangle = |n_x\rangle \otimes |n_y\rangle$ $(n_x, n_y = 0, 1, 2, \ldots)$ with eigenvalues

$$\mathbf{H}|n_x n_y\rangle = (\mathbf{H}_x|n_x\rangle) \otimes |n_y\rangle + |n_x\rangle \otimes (\mathbf{H}_y|n_y\rangle)$$
$$= \hbar\omega_0(n_x + n_y + 1)|n_x n_y\rangle.$$

We immediately note that the energy spectrum has become degenerate in 2D (assuming $\omega_x = \omega_y = \omega_0$), unlike the 1D case. Every combination of $n_x$ and $n_y$ which add to a given $N = n_x + n_y$ leads to the same energy eigenvalue $\hbar\omega_0(N+1)$. In general, there will be $N+1$ such combinations $(n_x = 0..N,\ n_y = N - n_x)$. Hence we cannot simply label the energy eigenstates by $N$, but must always provide two quantum numbers such as $|n_x n_y\rangle$, $|N n_x\rangle$, or $|N n_y\rangle$.

Looking ahead a bit, let's take this opportunity to introduce a different quantum number, which corresponds to *angular momentum*. In two kinetic dimensions, angular momentum is a scalar quantity. The corresponding observable is

$$\mathbf{L}_z = \mathbf{x}\mathbf{p}_y - \mathbf{y}\mathbf{p}_x,$$

which we label with a $z$ subscript to provide analogy with the 3D case. Note that operators on the $x$ and $y$ subspaces commute with each other, so the operator ordering in this definition is not critical.

As usual for a harmonic oscillator, our best way to proceed is to re-express the angular momentum operator in terms of annihilation and creation operators. Using

$$\mathbf{x} = \sqrt{\frac{\hbar}{2m\omega_0}}\left(\mathbf{a}_x + \mathbf{a}_x^\dagger\right), \quad \mathbf{p}_x = -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(\mathbf{a}_x - \mathbf{a}_x^\dagger\right),$$
$$\mathbf{y} = \sqrt{\frac{\hbar}{2m\omega_0}}\left(\mathbf{a}_y + \mathbf{a}_y^\dagger\right), \quad \mathbf{p}_y = -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(\mathbf{a}_y - \mathbf{a}_y^\dagger\right),$$

we obtain

$$\mathbf{L}_z = \frac{-i\hbar}{2}\left(\mathbf{a}_x + \mathbf{a}_x^\dagger\right)\left(\mathbf{a}_y - \mathbf{a}_y^\dagger\right) + \frac{i\hbar}{2}\left(\mathbf{a}_y + \mathbf{a}_y^\dagger\right)\left(\mathbf{a}_x - \mathbf{a}_x^\dagger\right)$$
$$= i\hbar\left(\mathbf{a}_x\mathbf{a}_y^\dagger - \mathbf{a}_x^\dagger\mathbf{a}_y\right).$$

Note that $\hbar$ thus appears as the natural unit of angular momentum! We can now easily verify that

$$[\mathbf{H}, \mathbf{L}_z] \propto \left[\mathbf{a}_x^\dagger\mathbf{a}_x + \mathbf{a}_y^\dagger\mathbf{a}_y, \mathbf{a}_x\mathbf{a}_y^\dagger - \mathbf{a}_x^\dagger\mathbf{a}_y\right]$$
$$= \left[\mathbf{a}_x^\dagger\mathbf{a}_x, \mathbf{a}_x\mathbf{a}_y^\dagger\right] - \left[\mathbf{a}_x^\dagger\mathbf{a}_x, \mathbf{a}_x^\dagger\mathbf{a}_y\right]$$
$$+ \left[\mathbf{a}_y^\dagger\mathbf{a}_y, \mathbf{a}_x\mathbf{a}_y^\dagger\right] - \left[\mathbf{a}_y^\dagger\mathbf{a}_y, \mathbf{a}_x^\dagger\mathbf{a}_y\right]$$
$$= \left[\mathbf{a}_x^\dagger\mathbf{a}_x, \mathbf{a}_x\right]\mathbf{a}_y^\dagger - \left[\mathbf{a}_x^\dagger\mathbf{a}_x, \mathbf{a}_x^\dagger\right]\mathbf{a}_y$$
$$+ \mathbf{a}_x\left[\mathbf{a}_y^\dagger\mathbf{a}_y, \mathbf{a}_y^\dagger\right] - \mathbf{a}_x^\dagger\left[\mathbf{a}_y^\dagger\mathbf{a}_y, \mathbf{a}_y\right]$$
$$= -\mathbf{a}_x\mathbf{a}_y^\dagger - \mathbf{a}_x^\dagger\mathbf{a}_y + \mathbf{a}_x\mathbf{a}_y^\dagger + \mathbf{a}_x^\dagger\mathbf{a}_y$$
$$= 0,$$

so the total energy and angular momentum can indeed be simultaneously be specified as quantum numbers for the 2D harmonic oscillator.

What about eigenstates for $\mathbf{L}_z$? The key here is to introduce new combinations of the $x$ and $y$ annihilation operators,

$$\mathbf{a}_L = \frac{1}{\sqrt{2}}(\mathbf{a}_x + i\mathbf{a}_y),$$
$$\mathbf{a}_R = \frac{1}{\sqrt{2}}(\mathbf{a}_x - i\mathbf{a}_y).$$

These definitions are motivated by analogy with optical polarizations, where we know we can express right- and left-circular polarization vectors as even superpositions of vertical and horizontal vectors with a relative phase of $e^{i\pi/2} = i.$ It is straightforward to verify

$$\left[\mathbf{a}_R, \mathbf{a}_R^\dagger\right] = \left[\mathbf{a}_L, \mathbf{a}_L^\dagger\right] = 1,$$

$$\left[\mathbf{a}_R, \mathbf{a}_L^\dagger\right] = \left[\mathbf{a}_L, \mathbf{a}_R^\dagger\right] = 0,$$

as we would want for annihilation and creation operators. Furthermore,

$$\mathbf{a}_L^\dagger \mathbf{a}_L = \frac{1}{2}\left(\mathbf{a}_x^\dagger \mathbf{a}_x + \mathbf{a}_y^\dagger \mathbf{a}_y - i\mathbf{a}_x^\dagger \mathbf{a}_y + i\mathbf{a}_x \mathbf{a}_y^\dagger\right),$$

$$\mathbf{a}_R^\dagger \mathbf{a}_R = \frac{1}{2}\left(\mathbf{a}_x^\dagger \mathbf{a}_x + \mathbf{a}_y^\dagger \mathbf{a}_y + i\mathbf{a}_x^\dagger \mathbf{a}_y - i\mathbf{a}_x \mathbf{a}_y^\dagger\right),$$

so

$$\mathbf{a}_R^\dagger \mathbf{a}_R + \mathbf{a}_L^\dagger \mathbf{a}_L = \mathbf{a}_x^\dagger \mathbf{a}_x + \mathbf{a}_y^\dagger \mathbf{a}_y$$

and we may write

$$\mathbf{H} = \hbar\omega_0\left(\mathbf{a}_R^\dagger \mathbf{a}_R + \mathbf{a}_L^\dagger \mathbf{a}_L + 1\right),$$

$$\mathbf{L}_z = i\hbar\left(\mathbf{a}_x \mathbf{a}_y^\dagger - \mathbf{a}_x^\dagger \mathbf{a}_y\right)$$

$$= \frac{\hbar}{2}(\mathbf{a}_R + \mathbf{a}_L)\left(\mathbf{a}_R^\dagger - \mathbf{a}_L^\dagger\right) - \frac{\hbar}{2}\left(\mathbf{a}_R^\dagger + \mathbf{a}_L^\dagger\right)(\mathbf{a}_L - \mathbf{a}_R)$$

$$= \frac{\hbar}{2}\left(\mathbf{a}_R \mathbf{a}_R^\dagger - \mathbf{a}_L \mathbf{a}_L^\dagger + \mathbf{a}_R^\dagger \mathbf{a}_R - \mathbf{a}_L^\dagger \mathbf{a}_L\right)$$

$$= \hbar\left(2\mathbf{a}_R^\dagger \mathbf{a}_R + 1 - 2\mathbf{a}_L^\dagger \mathbf{a}_L - 1\right)$$

$$= \hbar\left(\mathbf{a}_R^\dagger \mathbf{a}_R - \mathbf{a}_L^\dagger \mathbf{a}_L\right).$$

It follows from the above that we have new number states

$$|n_R n_L\rangle \propto \left(\mathbf{a}_R^\dagger\right)^{n_R}\left(\mathbf{a}_L^\dagger\right)^{n_L}|0_x\rangle \otimes |0_y\rangle,$$

with

$$\mathbf{H}|n_R n_L\rangle = \hbar\omega_0(n_R + n_L + 1)|n_R n_L\rangle$$

$$= \hbar\omega_0(N + 1)|n_R n_L\rangle$$

$$\mathbf{L}_z|n_R n_L\rangle = \hbar(n_R - n_L)|n_R n_L\rangle$$

$$\equiv m\hbar|n_R n_L\rangle.$$

Clearly the combinations $N \equiv n_R + n_L$ and $m \equiv n_R - n_L$ are linearly independent and thus uniquely specify an energy eigenstate. In particular, we see that for given $N,$ the angular momentum eigenvalue $m$ takes on values

$$m = -N, \ -N + 2, -N + 4, \ldots N - 4, N - 2, N.$$

(To see this, note that for fixed $N$ if we increase $n_R$ by one we must also decrease $n_L$ by one, hence the interval of $2$.) Thus the $(N + 1)$-fold degeneracy is completely lifted by $\mathbf{L}_z$, as specification of both $N$ and $m$ labels a unique $\left|n_R = \frac{N+m}{2},\, n_L = \frac{N-m}{2}\right\rangle.$

Finally, it is interesting to look at the wave functions corresponding to states $|N m\rangle.$ We clearly have

$$\mathbf{a}_x \leftrightarrow \sqrt{\frac{m\omega_0}{2\hbar}}\left(x + \frac{1}{m\omega_0}\frac{\partial}{\partial x}\right),$$
$$\mathbf{a}_y \leftrightarrow \sqrt{\frac{m\omega_0}{2\hbar}}\left(y + \frac{1}{m\omega_0}\frac{\partial}{\partial y}\right),$$

so

$$\mathbf{a}_R \leftrightarrow \sqrt{\frac{m\omega_0}{4\hbar}}\left(x - iy + \frac{1}{m\omega_0}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right),$$
$$\mathbf{a}_L \leftrightarrow \sqrt{\frac{m\omega_0}{4\hbar}}\left(x + iy + \frac{1}{m\omega_0}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right).$$

These can be re-expressed in cylindrical coordinates $(\rho, \varphi)$ as (this is a good exercise)

$$\mathbf{a}_R \leftrightarrow e^{i\varphi}\sqrt{\frac{m\omega_0}{4\hbar}}\left(\rho + \frac{1}{m\omega_0}\left(\frac{\partial}{\partial \rho} - \frac{i}{\rho}\frac{\partial}{\partial \varphi}\right)\right),$$
$$\mathbf{a}_L \leftrightarrow e^{i\varphi}\sqrt{\frac{m\omega_0}{4\hbar}}\left(\rho + \frac{1}{m\omega_0}\left(\frac{\partial}{\partial \rho} + \frac{i}{\rho}\frac{\partial}{\partial \varphi}\right)\right).$$

Solution of the resulting differential equations leads to the following [C-T *et al*, complement $D_{IV}$, Table I]. Note how the wave functions separate into radial and angular parts, with simple angular solutions $\exp(im\varphi)$ – this is a common feature for angular momentum eigenstates.

$n = 0$, $m = 0$: $$\chi_{0,0}(\rho) = \frac{\beta}{\sqrt{\pi}} e^{-\beta^2\rho^2/2}$$

$n = 1$, $m = 1$: $$\chi_{1,0}(\rho, \varphi) = \frac{\beta}{\sqrt{\pi}} \beta\rho\, e^{-\beta^2\rho^2/2} e^{i\varphi}$$

$n = 1$, $m = -1$: $$\chi_{0,1}(\rho, \varphi) = \frac{\beta}{\sqrt{\pi}} \beta\rho\, e^{-\beta^2\rho^2/2} e^{-i\varphi}$$

$n = 2$, $m = 2$: $$\chi_{2,0}(\rho, \varphi) = \frac{\beta}{\sqrt{2\pi}} (\beta\rho)^2 e^{-\beta^2\rho^2/2} e^{2i\varphi}$$

$n = 2$, $m = 0$: $$\chi_{1,1}(\rho, \varphi) = \frac{\beta}{\sqrt{\pi}} [(\beta\rho)^2 - 1] e^{-\beta^2\rho^2/2}$$

$n = 2$, $m = -2$: $$\chi_{0,2}(\rho, \varphi) = \frac{\beta}{\sqrt{2\pi}} (\beta\rho)^2 e^{-\beta^2\rho^2/2} e^{-2i\varphi}$$

### Orbital angular momentum, in general

Angular momentum operator in QM is defined just the same way as in classical mechanics:

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times \frac{\hbar}{i}\nabla,$$

where $\vec{\mathbf{r}} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $\vec{\mathbf{p}}$ are the 'vector' position and momentum operators. The cross-product notation has its usual meaning,

$$\mathbf{L}_x = \mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \quad \mathbf{L}_y = \mathbf{z}\mathbf{p}_x - \mathbf{x}\mathbf{p}_z, \quad \mathbf{L}_z = \mathbf{x}\mathbf{p}_y - \mathbf{y}\mathbf{p}_x.$$

Note that there are no subtleties here regarding operator ordering, since the product terms only involve operators acting on different 'kinetic' subspaces. Also, the angular momentum operators are clearly Hermitian.

Commutators of the angular momentum operators with position and momentum operators are easy to compute, *e.g.*,

$$[\mathbf{L}_x, \mathbf{x}] = [\mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \mathbf{x}] = 0,$$

$$[\mathbf{L}_x, \mathbf{y}] = [\mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \mathbf{y}] = -\mathbf{z}[\mathbf{p}_y, \mathbf{y}] = i\hbar\mathbf{z},$$

$$[\mathbf{L}_x, \mathbf{p}_y] = [\mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \mathbf{p}_y] = [\mathbf{y}, \mathbf{p}_y]\mathbf{p}_z = i\hbar\mathbf{p}_z,$$

etc. Hence, among the angular momentum operators themselves,

$$\begin{aligned}[\mathbf{L}_x, \mathbf{L}_y] &= [\mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \mathbf{z}\mathbf{p}_x - \mathbf{x}\mathbf{p}_z] \\ &= [\mathbf{y}\mathbf{p}_z, \mathbf{z}\mathbf{p}_x] + [\mathbf{z}\mathbf{p}_y, \mathbf{x}\mathbf{p}_z] \\ &= \mathbf{y}\mathbf{p}_x[\mathbf{p}_z, \mathbf{z}] + \mathbf{p}_y\mathbf{x}[\mathbf{z}, \mathbf{p}_z] \\ &= -i\hbar\mathbf{y}\mathbf{p}_x + i\hbar\mathbf{p}_y\mathbf{x} \\ &= i\hbar\mathbf{L}_z,\end{aligned}$$

and by cyclic permutation

$$[\mathbf{L}_x, \mathbf{L}_y] = i\hbar\mathbf{L}_z, \quad [\mathbf{L}_y, \mathbf{L}_z] = i\hbar\mathbf{L}_x, \quad [\mathbf{L}_z, \mathbf{L}_x] = i\hbar\mathbf{L}_y.$$

As we shall see over the next few lectures, this all-important set of commutation relations defines the structure of the angular momentum *algebra*. Note that this provides uncertainty relations of the form

$$\Delta\mathbf{L}_x\Delta\mathbf{L}_y \geq \frac{\hbar}{2}|\langle\mathbf{L}_z\rangle|,$$

and so on.

## Algebraic derivation of eigenvalues

Similar to the LHO, the commutation relations can be used as the basis of an algebraic approach to solving the angular momentum eigenvalue problem. Following common convention we switch notation in this section to $\vec{\mathbf{J}} \leftrightarrow \vec{\mathbf{L}}$,

$$[\mathbf{J}_x, \mathbf{J}_y] = i\hbar\mathbf{J}_z, \quad [\mathbf{J}_y, \mathbf{J}_z] = i\hbar\mathbf{J}_x, \quad [\mathbf{J}_z, \mathbf{J}_x] = i\hbar\mathbf{J}_y,$$

in order to emphasize that the derivation applies not only to orbital angular momentum but any set of operators (*e.g.* spin) that satisfy the given commutation relations.

First we define a few new operators in terms of the old ones,

$$\mathbf{J}^2 \equiv \vec{\mathbf{J}} \cdot \vec{\mathbf{J}} = \mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2,$$

$$\mathbf{J}_+ \equiv \mathbf{J}_x + i\mathbf{J}_y,$$

$$\mathbf{J}_- = \mathbf{J}_+^\dagger = \mathbf{J}_x - i\mathbf{J}_y.$$

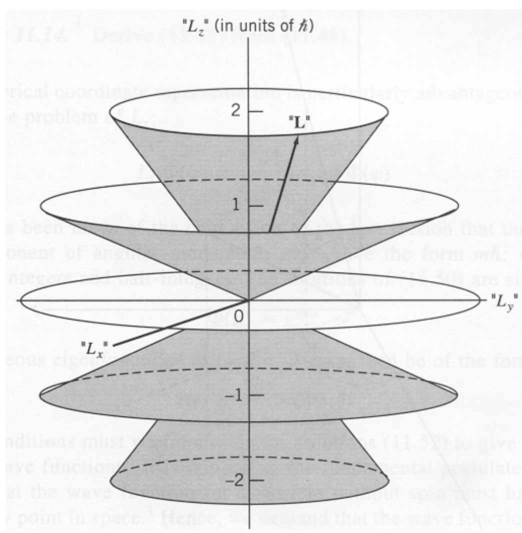
As you might guess from the notation, $\mathbf{J}_{\pm}$ will play similar roles to those of the annihilation and creation operators. These angular momentum raising and lowering operators are clearly not Hermitian, but the 'total angular momentum' operator $\mathbf{J}^2$ is Hermitian. Note that

$$\begin{aligned}
[\mathbf{J}^2, \mathbf{J}_x] &= [\mathbf{J}_y^2, \mathbf{J}_x] + [\mathbf{J}_z^2, \mathbf{J}_x] \\
&= \mathbf{J}_y^2\mathbf{J}_x - \mathbf{J}_x\mathbf{J}_y^2 + \mathbf{J}_z^2\mathbf{J}_x - \mathbf{J}_x\mathbf{J}_z^2 \\
&= \mathbf{J}_y^2\mathbf{J}_x - (\mathbf{J}_y\mathbf{J}_x + i\hbar\mathbf{J}_z)\mathbf{J}_y + \mathbf{J}_z(\mathbf{J}_x\mathbf{J}_z + i\hbar\mathbf{J}_y) - \mathbf{J}_x\mathbf{J}_z^2 \\
&= \mathbf{J}_y^2\mathbf{J}_x - \mathbf{J}_y\mathbf{J}_x\mathbf{J}_y - i\hbar\mathbf{J}_z\mathbf{J}_y + \mathbf{J}_z\mathbf{J}_x\mathbf{J}_z + i\hbar\mathbf{J}_z\mathbf{J}_y - \mathbf{J}_x\mathbf{J}_z^2 \\
&= \mathbf{J}_y^2\mathbf{J}_x - \mathbf{J}_y(\mathbf{J}_y\mathbf{J}_x + i\hbar\mathbf{J}_z) + (\mathbf{J}_x\mathbf{J}_z + i\hbar\mathbf{J}_y)\mathbf{J}_z - \mathbf{J}_x\mathbf{J}_z^2 \\
&= -i\hbar\mathbf{J}_y\mathbf{J}_z + i\hbar\mathbf{J}_y\mathbf{J}_z \\
&= 0,
\end{aligned}$$

and

$$[\mathbf{J}^2, \mathbf{J}_x] = [\mathbf{J}^2, \mathbf{J}_y] = [\mathbf{J}^2, \mathbf{J}_z] = 0$$

by symmetry. Hence, we find that while there are no states that have zero uncertainty for more than one component of $\vec{\mathbf{J}}$ (except those with $\langle \vec{\mathbf{J}} \rangle = 0$), there should exist simultaneous eigenstates of $\mathbf{J}^2$ and any one Cartesian component of angular momentum. If we think about this in classical terms, it's like saying that we can't precisely define more than one Cartesian component of the angular momentum vector, but we can precisely define its *length* together with any one Cartesian component. This line of reasoning leads to the 'semiclassical' picture of angular momentum [Merzbacher 11.2],




From this picture, we may also infer that for a given eigenvalue of $\mathbf{J}^2$, the corresponding eigenvalues of $\mathbf{J}_{x,y,z}$ must lie within a certain range.

Let's label the simultaneous eigenstates of $\mathbf{J}^2$ and $\mathbf{J}_z$ (for instance) by $|\lambda m\rangle$, with

$$\mathbf{J}^2|\lambda m\rangle = \lambda\hbar^2|\lambda m\rangle, \quad \mathbf{J}_z|\lambda m\rangle = m\hbar|\lambda m\rangle,$$

where we are making use of our previous insight that $\hbar$ is the natural unit of angular momentum. According to the semiclassical picture, we would like to think that $m^2 \leq \lambda$. To see that this is indeed true, we may write

$$\mathbf{J}^2 - \mathbf{J}_z^2 = \mathbf{J}_x^2 + \mathbf{J}_y^2 = \frac{1}{4}\left[(\mathbf{J}_+ + \mathbf{J}_-)^2 - (\mathbf{J}_+ - \mathbf{J}_-)^2\right] = \frac{1}{2}(\mathbf{J}_+\mathbf{J}_- + \mathbf{J}_-\mathbf{J}_+) = \frac{1}{2}\left(\mathbf{J}_+\mathbf{J}_+^\dagger + \mathbf{J}_+^\dagger\mathbf{J}_+\right).$$

Recall that the expectation value of an operator of the form $\mathbf{A}\mathbf{A}^\dagger$ or $\mathbf{A}^\dagger\mathbf{A}$ must be non-negative since, *e.g.*,

$$\left\langle\Psi|\mathbf{A}^\dagger\mathbf{A}|\Psi\right\rangle = (\mathbf{A}|\Psi\rangle)^\dagger(\mathbf{A}|\Psi\rangle) = |\mathbf{A}|\Psi\rangle|^2 \geq 0.$$

Hence we conclude that

$$\langle\Psi|\mathbf{J}^2 - \mathbf{J}_z^2|\Psi\rangle = \frac{1}{2}\left\langle\Psi|\mathbf{J}_+\mathbf{J}_+^\dagger|\Psi\right\rangle + \frac{1}{2}\left\langle\Psi|\mathbf{J}_+^\dagger\mathbf{J}_+|\Psi\right\rangle \geq 0$$

for any state, and in particular for simultaneous eigenstates $|\lambda m\rangle$,

$$\langle\lambda m|\mathbf{J}^2 - \mathbf{J}_z^2|\lambda m\rangle = \lambda\hbar^2 - m^2\hbar^2 \geq 0, \quad \lambda \geq m^2.$$

For a given value of $\lambda$, we have thus established an allowed range for $m$. This may remind us of the procedure we used in the case of the LHO, where we found a lower bound on eigenvalues of the number operator and this gave us a starting point for deriving its eigenspectrum. Motivated by this analogy, we would next like to show that the angular momentum raising and lowering operators can be used to obtain new eigenstates from old eigenstates:

$$\begin{aligned}
\mathbf{J}_z|\lambda m\rangle &= m\hbar|\lambda m\rangle, \\
\mathbf{J}_z[\mathbf{J}_+|\lambda m\rangle] &= \mathbf{J}_z(\mathbf{J}_x + i\mathbf{J}_y)|\lambda m\rangle \\
&= (\mathbf{J}_x\mathbf{J}_z + i\hbar\mathbf{J}_y + i(\mathbf{J}_y\mathbf{J}_z - i\hbar\mathbf{J}_x))|\lambda m\rangle \\
&= [(\mathbf{J}_x + i\mathbf{J}_y)\mathbf{J}_z + \hbar(\mathbf{J}_x + i\mathbf{J}_y)]|\lambda m\rangle \\
&= (m+1)\hbar[\mathbf{J}_+|\lambda m\rangle], \\
\mathbf{J}_z[\mathbf{J}_-|\lambda m\rangle] &= \mathbf{J}_z(\mathbf{J}_x - i\mathbf{J}_y)|\lambda m\rangle = (m-1)\hbar[\mathbf{J}_-|\lambda m\rangle].
\end{aligned}$$

Also, $\mathbf{J}_\pm$ commutes with $\mathbf{J}^2$ since $\mathbf{J}_{x,y}$ do, so

$$\mathbf{J}^2[\mathbf{J}_\pm|\lambda m\rangle] = \mathbf{J}_\pm[\mathbf{J}^2|\lambda m\rangle] = \lambda\hbar^2[\mathbf{J}_\pm|\lambda m\rangle].$$

Hence either $\mathbf{J}_\pm|\lambda m\rangle = 0$ or $\mathbf{J}_\pm|\lambda m\rangle$ is *proportional to* a new simultaneous eigenstate $|\lambda(m\pm 1)\rangle$:

$$\begin{aligned}
\mathbf{J}_+|\lambda m\rangle &= C_+(\lambda, m)\,\hbar|\lambda(m+1)\rangle, \\
\mathbf{J}_-|\lambda m\rangle &= C_-(\lambda, m)\,\hbar|\lambda(m-1)\rangle,
\end{aligned}$$

where $C_\pm(\lambda, m)$ are normalization coefficients (which may be complex, in principle) yet to be determined.

Now we use the fact that there are lower and upper bounds on $m$ (in terms of $\lambda$). Using $j_\lambda$ to denote the maximum value of $m$ for a given $\lambda$, we have

$$\mathbf{J}_+|\lambda j_\lambda\rangle = 0.$$

We would like to use this 'boundary condition' to derive $j_\lambda$ in terms of $\lambda$, so let's try to put $\mathbf{J}^2$ and $\mathbf{J}_z$ on the LHS. An easy way to do this is to multiply from the left by $\mathbf{J}_-$, yielding

$$\begin{aligned} 0 &= \mathbf{J}_-\mathbf{J}_+|\lambda j_\lambda\rangle \\ &= (\mathbf{J}_x - i\mathbf{J}_y)(\mathbf{J}_x + i\mathbf{J}_y)|\lambda j_\lambda\rangle \\ &= (\mathbf{J}_x^2 + \mathbf{J}_y^2 + i\mathbf{J}_x\mathbf{J}_y - i\mathbf{J}_y\mathbf{J}_x)|\lambda j_\lambda\rangle \\ &= (\mathbf{J}^2 - \mathbf{J}_z^2 - \hbar\mathbf{J}_z)|\lambda j_\lambda\rangle \\ &= (\lambda - j_\lambda^2 - j_\lambda)\hbar^2|\lambda j_\lambda\rangle. \end{aligned}$$

Hence we have

$$\begin{aligned} \lambda - j_\lambda^2 - j_\lambda &= 0, \\ \lambda &= j_\lambda(j_\lambda + 1). \end{aligned}$$

Similarly, on the lower end we have

$$\mathbf{J}_-|\lambda j'_\lambda\rangle = 0,$$

where $j'_\lambda$ denotes the minimum value of $m$ for a given $\lambda.$ This leads to

$$\begin{aligned} 0 &= \mathbf{J}_+\mathbf{J}_-|\lambda j'_\lambda\rangle \\ &= (\mathbf{J}_x^2 + \mathbf{J}_y^2 - i\mathbf{J}_x\mathbf{J}_y + i\mathbf{J}_y\mathbf{J}_x)|\lambda j'_\lambda\rangle \\ &= (\mathbf{J}^2 - \mathbf{J}_z^2 + \hbar\mathbf{J}_z)|\lambda j'_\lambda\rangle \\ &= \left(\lambda - (j'_\lambda)^2 + j'_\lambda\right)\hbar^2|\lambda j'_\lambda\rangle, \end{aligned}$$

and

$$\lambda = j'_\lambda(j'_\lambda - 1).$$

The two conditions we have derived yield a consistency condition

$$j_\lambda(j_\lambda + 1) = j'_\lambda(j'_\lambda - 1),$$

whose solutions are

$$j'_\lambda = -j_\lambda, \qquad j'_\lambda = j_\lambda + 1.$$

The latter solution is not allowed since we have defined $j_\lambda \geq j'_\lambda$, so we may henceforth assume that $j'_\lambda = -j_\lambda.$

It is important to note that the bounds $j_\lambda, j'_\lambda$ we have derived are exact, since for example $\mathbf{J}_-|\lambda m\rangle = 0$ is satisfied only by $m = j'_\lambda.$ Suppose we now start with the lowest eigenstate $|\lambda j'_\lambda\rangle$ and repeatedly apply the raising operator $\mathbf{J}_+.$ We saw above that this leads (up to normalization) to new eigenstates

$$(\mathbf{J}_+)^n|\lambda j'_\lambda\rangle \propto |\lambda(j'_\lambda + n)\rangle$$

until the upper bound on $m$ is exceeded. Again, this ladder will truncate properly only if ends precisely at $|\lambda j_\lambda\rangle,$ since $\mathbf{J}_+|\lambda m\rangle = 0$ is satisfied only by $m = j_\lambda.$ Hence we conclude that $j_\lambda = j'_\lambda + n = -j_\lambda + n$ for some integer $n \geq 0,$ or equivalently

$$2j_\lambda = n.$$

This shows that $j_\lambda$ can only be a non-negative integer or half-integer. For a given $\lambda,$ and thus given $j_\lambda,$ we see that the possible eigenvalues of $J_z$ are

$$m\hbar = j_\lambda\hbar,\ (j_\lambda - 1)\hbar,\ (j_\lambda - 2),\ \cdots - (j_\lambda - 1)\hbar,\ \ -j_\lambda\hbar$$

are likewise all integral multiples of $\hbar$ (for integer $j_\lambda$) or half-integral multiples of $\hbar$ (for half-integer $j_\lambda$). In general, there are $2j_\lambda + 1$ allowed values of $m$ for a given $\lambda.$ Hence, we may think of the eigenstates $|j;m\rangle$ as being arranged in a 'tiered' structure:

| |
|---|
| $\|0;0\rangle$ |
| $\|1/2;-1/2\rangle \quad \|1/2;1/2\rangle$ |
| $\|1;-1\rangle \quad \|1;0\rangle \quad \|1;1\rangle$ |
| $\|3/2;-3/2\rangle \quad \|3/2;-1/2\rangle \quad \|3/2;1/2\rangle \quad \|3/2;3/2\rangle$ |
| $\|2;-2\rangle \quad \|2;-1\rangle \quad \|2;0\rangle \quad \|2;1\rangle \quad \|2;2\rangle$ |
| $\vdots$ |

Up to this point we have thought of $\lambda$ and $m$ as the natural quantum numbers to specify angular momentum eigenstates, since we started by considering simultaneous eigenstates of $\mathbf{J}^2$ and $\mathbf{J}_z$. Noting that $\lambda$ and $j_\lambda$ are in one-to-one correspondence, however, we can just as well label eigenstates according to $m$ and $j,$ with $\lambda = j(j+1).$ This is in fact the more common convention, where $j$ is often referred to as 'the angular momentum' of a state $|jm\rangle$ and $m$ is its 'azimuthal quantum number.'

From this point of view, quantum mechanics gives us the following picture:

1. Angular momentum is quantized. The angular momentum state space breaks down into subspaces of fixed $j$, where $j$ must be an integer or half-integer (including zero). The subspace with angular momentum $j$ has $2j+1$ basis states.
2. No two Cartesian components of the angular momentum vector $\langle\vec{J}\rangle$ can be simultaneously specified without uncertainty.
3. Any one Cartesian component $m\hbar$ may be precisely specified together with $j.$
4. The 'length' of the angular momentum vector $\sqrt{\langle\mathbf{J}^2\rangle} = \hbar\sqrt{j(j+1)}$ corresponds to $\hbar j$ only in the limit $j \to \infty.$
5. The length $\hbar\sqrt{j(j+1)}$ is therefore always greater than any single Cartesian component $\hbar m,$ with the difference made up by 'fluctuations' in the orthogonal components.

Finally, let us compute the normalization coefficients $C_\pm(\lambda,m),$ defined above. This will allow us to contemplate matrix representations of the angular momentum operators. For normalization we require

$$||\lambda(m\pm 1)\rangle|^2 = \left[\hbar^2|C_\pm(\lambda,m)|^2\right]^{-1}(\mathbf{J}_\pm|\lambda m\rangle)^\dagger(\mathbf{J}_\pm|\lambda m\rangle) = 1,$$

$$\begin{aligned}\hbar^2|C_\pm(\lambda,m)|^2 &= \langle\lambda m|\mathbf{J}_\mp\mathbf{J}_\pm|\lambda m\rangle \\ &= \langle\lambda m|(\mathbf{J}_x \mp i\mathbf{J}_y)(\mathbf{J}_x \pm i\mathbf{J}_y)|\lambda m\rangle \\ &= \langle\lambda m|\mathbf{J}^2 - \mathbf{J}_z^2 \mp \hbar\mathbf{J}_z|\lambda m\rangle \\ &= (\lambda - m^2 \mp m)\hbar^2.\end{aligned}$$

Hence we may set

$$
\begin{aligned}
|C_\pm(\lambda,m)|^2 &= \lambda - m^2 \mp m \\
&= j(j+1) - m^2 \mp m \\
&= j(j+1) - m(m \pm 1).
\end{aligned}
$$

It is customary to set the phase of these coefficients to zero. We may then write (with a little rearranging)

$$
\begin{aligned}
C_+(\lambda,m) &= \sqrt{(j-m)(j+m+1)}\,, \\
C_-(\lambda,m) &= \sqrt{(j+m)(j-m+1)}\,.
\end{aligned}
$$

We thus see that, within a given $j$ subspace, the raising and lowering operators may be represented as $(2j+1)$-dimensional matrices with real nonzero elements only on the first super- or sub-diagonal. Since all the angular momentum operators can in fact be given as linear combinations of the raising and lowering operators, we may conclude that the $j$ subspaces are in fact closed with respect to operation of the angular momentum operators. As we'll discuss in coming lectures, this important insight will help us understand addition of angular momenta in terms of linear representations of the angular momentum algebra.