7.1.2. The Space-Time as Development Information of Extension Relation is Called $\mathfrak{D}$-space

What classically is called Space in physics we in this book call a $\mathfrak{Z}$-space ***quality***, and we have developed the Geometric Algebra $\mathcal{G}_3(\mathbb{R})$ concerning four ***grades*** of the traditional three dimensions. To make measurements of the extensions we need a signal of information transmitted over that physical space extension. We have an *a priori synthetic judgment* that the speed of information development is isotropic finite relative to all physical ***directions*** in $\mathfrak{Z}$-space.
The development ***direction*** into the future is obvious *total independent* of the space extensions, thus *orthogonal* to these. The ***quantitative*** measure of development ***quality*** with a parameter is first in this book introduced in § 1.4.1.1, etc. as a resulting real continuous monotonously growing *development parameter* $t_c \in \vec{\mathbb{R}}$, and further ***quantified*** through chapter 3 as a carrier clock.

- In the classical tradition the impact of this *development parameter* is called **Time**.

Relations between ***entities*** over physical **Space** extension depend on the measurement of the magnitude of extension separation, don by a (light) signal with the speed of information c.
Traditionally this extension measure relation is simply expressed as $|\mathbf{x}_{AB}| = c|t_B - t_A| \in \mathbb{R}$.
Therefore, we simply have the traditional expression **Space-Time** for the information dependent development relations in $\mathfrak{D}$-space between the extension separated ***entities*** in $\mathfrak{Z}$-space.

7.1.3. The Full Geometric Space-Time Algebra $\mathcal{G}_{1,3}(\mathbb{R})$ for Physical Relations in $\mathfrak{D}$-space

For a general multivector element $X \in \mathcal{G}_{1,3}(\mathbb{R})$ as the geometric Space-Time algebra (STA) we can separate into five different ***grades*** of the ***direction*** ideas for a now known $\mathfrak{D}$-space

(7.9) $$X = \langle X\rangle_0 + \langle X\rangle_1 + \langle X\rangle_2 + \langle X\rangle_3 + \langle X\rangle_4 \;=\; \textstyle\sum_{r=0}^{4}\langle X\rangle_r \quad \in \mathcal{G}_{1,3}(\mathbb{R})$$

We, humans, define this STA algebra from the traditional mathematical idea of a 1-vector space over the real number field $\mathbb{R}^4 \to V_{1,3}(\mathbb{R})$ with a $\mathfrak{D}$-space physical ***directional*** founded orthonormal unit basis $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, that is chosen with a Minkowski metric Clifford signature $(+,-,-,-)$ where (7.3)-(7.4) are valid. The linear 1-vector space $V_{1,3}(\mathbb{R})$ fulfil the additional algebraic rules (4.1)-(4.11) and the linear combination represents every possible 1-vector[405]

(7.10) $$\langle X\rangle_1 = x \;=\; x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3 \;= \textstyle\sum_{\mu=0}^{3} x^\mu\gamma_\mu \quad \in V_{1,3}(\mathbb{R}) \subset \mathcal{G}_{1,3}(\mathbb{R}).$$

Now we form the full multiplication algebra $\mathcal{G}_{1,3}(\mathbb{R})$ following the associative and left or right distributive rules (5.38)-(5.42) with the maximum of ***fourth grade*** for the multiplications.
The top ***grade four*** $\langle X\rangle_4$ is the helicity pseudoscalar ***direction quality*** in STA. (a 4D $\mathfrak{D}$-volume)
For this ***primary quality of fourth grade direction*** (***pqg***-4) in $\mathfrak{D}$-space of physics, we define by orthonormality of the outer product *the unit dextral helicity pseudoscalar*

(7.11) $$\boxed{\imath := \gamma_1\gamma_2\gamma_3\gamma_0} \;=\; \gamma_1\wedge\gamma_2\wedge\gamma_3\wedge\gamma_0 \;=\; \gamma_0\wedge\gamma_3\wedge\gamma_2\wedge\gamma_1 \;=\; \gamma_0\gamma_3\gamma_2\gamma_1. \qquad (7.8)$$ [406]

For all possible elements X in consideration, we demand $\imath\wedge X = \gamma_3\wedge\gamma_2\wedge\gamma_1\wedge\gamma_0\wedge X = 0$ to be in STA

(7.12) $$\forall X \in \mathcal{G}_{1,3}(\mathbb{R}) \;\Leftrightarrow\; \imath\wedge X = 0 = X\wedge\imath, \quad \text{or } \langle X\rangle_5 = 0, \text{ etc., } \textbf{grades pqg-}r \le 4, \text{ exeption } X = \langle X\rangle_0 \in \mathbb{R}.$$

Further by orthonormality $\gamma_\nu\gamma_\mu = \gamma_\nu\wedge\gamma_\mu$, we note the anti-commutation of the six basic *bivectors*

(7.13) $$\gamma_\nu\gamma_\mu = -\gamma_\mu\gamma_\nu, \quad \text{for } \mu \ne \nu,\ \mu,\nu = 0,1,2,3,$$ the ***direction of second grade*** (***pqg***-2) in $\mathfrak{D}$-space.

And we can form four basis *trivectors* ***directions***, that have reversed orientations too:

(7.14) $$\gamma_1\gamma_2\gamma_3 = -\gamma_3\gamma_2\gamma_1, \quad \gamma_2\gamma_3\gamma_0 = -\gamma_0\gamma_3\gamma_2, \quad \gamma_3\gamma_1\gamma_0 = -\gamma_0\gamma_1\gamma_3, \quad \gamma_1\gamma_2\gamma_0 = -\gamma_0\gamma_2\gamma_1.$$

These ***primary quality of third grade direction*** (***pqg***-3) is dual to those ***of first grade*** (***pqg***-1)

(7.15) $$\imath\gamma_0 = \gamma_1\gamma_2\gamma_3, \qquad \imath\gamma_1 = \gamma_2\gamma_3\gamma_0, \qquad \imath\gamma_2 = \gamma_3\gamma_1\gamma_0, \qquad \imath\gamma_3 = \gamma_1\gamma_2\gamma_0.$$

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[405] We here use contravariant[289] coordinates $x^\mu \in\mathbb{R}$ due to mixed signature (7.25)↓, and italic x^μ in stet of Greek $\lambda^\mu \in\mathbb{R}$ for the field. Remark that we no longer use bold letters for the elements e.g., $X, x \in \mathcal{G}_{1,3}(\mathbb{R})$. This included the reals $x_\mu \in\mathbb{R} \subset \mathcal{G}_{1,3}(\mathbb{R})$ in STA.

[406] Remark the difference to the Hestenes [6] etc., definition $i = \gamma_0\gamma_1\gamma_2\gamma_3 = -\imath$, due to sequence $\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ used in this book. (reversed)

What physical ***directional quality*** dos these four trivectors represent in $\mathfrak{D}$-space?
The first $\imath\gamma_0 = -\gamma_0\imath$ represent the dextral trivector structure ***direction*** of STA relative to the physical *active helicity pseudoscalar* ***directional primary quality of grade four*** in STA geometry.[407]
The three others represent the orthogonal angular momenta ***directions*** of the possible active cyclic oscillations in space like extension planes, e.g., see Table 5.3. The connection interaction products of two of these trivectors just give a bivector ***quality***, e.g. $\imath\gamma_2\imath\gamma_1 = \gamma_1\gamma_2 \sim \boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3$.
Special for the six orthonormal bivectors ***of second grade*** (7.13) we have the dual relations

(7.16) $$\begin{array}{llll} \gamma_2\gamma_3 = \imath\gamma_1\gamma_0, & \gamma_3\gamma_1 = \imath\gamma_2\gamma_0, & \gamma_1\gamma_2 = \imath\gamma_3\gamma_0 & \in \mathcal{G}_{1,3}(\mathbb{R}) \sim \mathcal{G}_4(\mathbb{R}). \\ \sim \boldsymbol{i}_1 = \boldsymbol{i}\boldsymbol{\sigma}_1, & \sim \boldsymbol{i}_2 = \boldsymbol{i}\boldsymbol{\sigma}_2, & \sim \boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3 & \in \mathcal{G}_3(\mathbb{R}). \end{array} \updownarrow$$

Now we know all ***four directional grades***. We also have the ***non-directional zero grade*** real scalar field $\mathbb{R}$ founding the span of the full linear space of $\mathcal{G}_{1,3}(\mathbb{R})$ as the geometric algebra STA.
We use the general definition of the inner product (5.57) on the 1-vector basis $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$

(7.17) $$g_{\mu\nu} = \gamma_\mu\cdot\gamma_\nu = \tfrac{1}{2}\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\right) = \gamma_\nu\cdot\gamma_\mu \in\mathbb{R}, \quad \text{for } \mu = 0,1,2,3, \quad \text{each a real scalar: } 1, -1 \text{ or } 0.$$

This is called for the *metric tensor* for the orthonormal *standard frame* $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ for STA,

(7.18) $$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ for orthonormality, and } \; g_{\mu\mu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \text{ for the normalized metric.}$$

For the orthogonal situation for $\mu \ne \nu$ we have the appreciated anti-commutator relation:

(7.19) $$\boxed{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0}.$$

This not only expresses the algebraic linear independency but also gives the possibility to manage the implicit hidden interdependent *connectivity* between the ***direction qualities*** of physics.
The unit metric for the standard frame $\{\gamma_\mu\,;\ \mu = 0,1,2,3\}$ is (with no $\mu\mu$ sum)

(7.20) $$g_{\mu\mu} = \gamma_\mu\cdot\gamma_\mu = \gamma_\mu\gamma_\mu = \gamma_\mu^2 \;\Rightarrow (\gamma_0^2, \gamma_1^2, \gamma_2^2, \gamma_3^2) = (1, -1, -1, -1) \Rightarrow \text{signature } (+,-,-,-).$$

Here we recall the most fundamental concept of making measurements in physics is the unit count named γ_0 for the development *causal* ***direction*** *FORWARD*. This is independent $\overline{\gamma_0} = \gamma_0$ (5.300) - of the Descartes extension *parity inversion* conjugation $\overline{\gamma_k} = -\gamma_k$ (5.301) for the three ***directions***
For each of these three perpendicular *isometric* ***directions***, we demand the measure balance (5.302)

(7.21) $$\boxed{\gamma_0^2 + \gamma_k^2 = 0} \quad k = 1,2,3.$$

This is the absolute fundamental relative geometrical information measure relation of physics.
The multiplicative inverse of a multivector component is defined in (4.76) $x^{-1} = x/x^2$ fulfilling

(7.22) $$x \cdot x^{-1} = 1 \quad \Rightarrow \quad \gamma_\mu \cdot \gamma_\mu^{-1} = 1, \qquad \text{hence:}$$

We call the multiplicative inverse γ_μ^{-1} the reciprocal basis 1-vector and rename it $\gamma^\mu := \gamma_\mu^{-1}$.
By this we achieve the reciprocal basis frame $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ defined through the relation

(7.23) $$\gamma_\mu\cdot\gamma^\nu = \gamma^\nu\cdot\gamma_\mu = \delta_\mu^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{or the transformations} \quad \gamma_\mu = g_{\mu\nu}\gamma^\nu, \quad \gamma^\nu = g^{\mu\nu}\gamma_\mu.$$

The idea is so simple that the two reciprocal basis is *extension parity inversion* of each other

(7.24) $$\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\} = \overline{\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}} = \{\gamma_0, -\gamma_1, -\gamma_2, -\gamma_3\}, \qquad \text{measure } \gamma^0 = \gamma_0 \text{ is covariant.}$$

We remark that $\{\gamma^\mu\}$ also has the signature $(+,-,-,-) \Leftarrow ({\gamma^0}^2, {\gamma^1}^2, {\gamma^2}^2, {\gamma^3}^2) = (1, -1, -1, -1)$.

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[407] We use two different nomenclature for the static eternal dextral pseudoscalar $\boldsymbol{i} \in \mathcal{G}_3(\mathbb{R})$ of $\mathfrak{Z}$-space and the active helicity pseudoscalar $\imath \in \mathcal{G}_{1,3}(\mathbb{R}) \sim \mathcal{G}_4(\mathbb{R})$. These semiotic different symbols $\boldsymbol{i}$ and $\imath$ refer to different physical ***qualities*** of Nature. In our ethical work of physics, we shall distinguish these. In traditional aesthetics of mathematics, the idea of $\sqrt{-1}$ has been one monotheistic imaginary unit $i = \mathrm{Im}(1) \Leftarrow \sqrt{-1}$ with $i \in \mathbb{C}$ of the complex number field. But in the geometry of physics, we are forced to distinguish the different ***grades of direction*** although we have the similarity: $\left(\sqrt{-1}\right)^2 = i^2 = \mathbf{i}^2 = \boldsymbol{i}^2 = \imath^2 = -1$.
It is obvious that our universal nature is not that fragmented, so the connected inherence is written $\imath \rightleftarrows \boldsymbol{i} \rightleftarrows \mathbf{i} \rightleftarrows i \rightleftarrows \sqrt{-1}$.

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