The autonomy of one ***entity*** $\Psi_{½}$ makes it possible to choose any one of these frame ***directions*** $\mathbf{u}_\mu = \mathbf{u}$ for the projection $P_{\mathbf{u}_\mu} = P_{\mathbf{u}}$ internal in any spin½ fermion ***entity*** $\Psi_{½}$ in physical 3-space.
The foundation of this picture is the four internal orientations of the cyclic oscillations in the plane ***directions*** $\boldsymbol{i}\mathbf{u}_\mu$ of the regular tetrahedron four faces, around the ***outwards*** 1-vector ***directions*** $\mathbf{u}_\mu$ of the regular tetraon fulfilling (6.460) expressing the *symmetry*

(6.481) $$\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3=0 \Leftrightarrow \mathbf{s}_0+\mathbf{s}_1+\mathbf{s}_2+\mathbf{s}_3=0, \quad \text{and} \quad \mathbf{u}_\mu\cdot\mathbf{u}_\nu = -\tfrac{1}{3}, \text{ for } \mu\neq\nu \quad (6.460), \quad \mathbf{s}_\mu = \tfrac{1}{2}\mathbf{u}_\mu.$$

When we change all four orientations in the tetraon symmetry we instead get one positive quantity

(6.482) $$q = (+\mathbf{s}_0-\mathbf{s}_1-\mathbf{s}_2-\mathbf{s}_3)\mathbf{u}_0 = 1. \qquad \text{(just as a charge of a positron).}$$

When all orientations are the same, we achieve a completely neutral indivisible fermion ***entity*** $\Psi_{½}$

(6.483) $$q = (-\mathbf{s}_0-\mathbf{s}_1-\mathbf{s}_2-\mathbf{s}_3)\mathbf{u}_3 = (+\mathbf{s}_0+\mathbf{s}_1+\mathbf{s}_2+\mathbf{s}_3)\mathbf{u}_3 = 0.$$

This neutral spin½ ***entity quality*** carries no ***quantitative*** cargo. (just as an uncharged neutrino).
We will not in this volume go into the impact of the fermions $\Psi_{½}$ with fractional q quantities and how these indivisible ***entities*** can exclusively combine. The reader is encouraged to do this work.

6.5.10.6. The Freedom of the Tetraon Angular Composition in one Fermion

In our humanistic mathematical approach to physics, we have the freedom to model in lots of ways.
To versatile this from (6.461) we look at the resulting regular tetraon structure of the scheme in (6.444), noting that the magnitudes of the eight (two times four) total angular momenta all equal

(6.484) $$\left|\mathbf{L}^{\Psi_{½}}_{\mathrm{s}...}\right| = \left|\mathbf{L}^{\Psi_{½}}_{\mathrm{d}...}\right| = \left|\mathbf{j}^{\Psi_{½}}_{\mathrm{s}...}\right| = \left|\mathbf{j}^{\Psi_{½}}_{\mathrm{d}...}\right| = \sqrt{¾} = \sqrt{J^2}$$

The reader should remark that this has tetrahedron edge length $\sqrt{2}$ just as $|\pm\boldsymbol{\sigma}_1\pm\boldsymbol{\sigma}_2| = \sqrt{2}$ etc.
We note that we have autonomously presumed $|\mathbf{L}_1| = |\mathbf{L}_2| = |\mathbf{L}_3| = ½$.
Combining these in a dextral way and multiplying with $\frac{2}{\sqrt{3}}$ we get the four unit 1-vector basis

(6.485) $$\begin{aligned}
\mathbf{u}_0 &= \tfrac{2}{\sqrt{3}}\,\mathbf{j}^{\Psi_{½}}_{\mathrm{d}+++} = \tfrac{2}{\sqrt{3}}(+\mathbf{j}_1+\mathbf{j}_2+\mathbf{j}_3) = \sqrt{⅓}\,(+\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3),\\
\mathbf{u}_1 &= \tfrac{2}{\sqrt{3}}\,\mathbf{j}^{\Psi_{½}}_{\mathrm{d}+--} = \tfrac{2}{\sqrt{3}}(+\mathbf{j}_1-\mathbf{j}_2-\mathbf{j}_3) = \sqrt{⅓}\,(+\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_3),\\
\mathbf{u}_2 &= \tfrac{2}{\sqrt{3}}\,\mathbf{j}^{\Psi_{½}}_{\mathrm{d}-+-} = \tfrac{2}{\sqrt{3}}(-\mathbf{j}_1+\mathbf{j}_2-\mathbf{j}_3) = \sqrt{⅓}\,(-\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_3),\\
\mathbf{u}_3 &= \tfrac{2}{\sqrt{3}}\,\mathbf{j}^{\Psi_{½}}_{\mathrm{d}--+} = \tfrac{2}{\sqrt{3}}(-\mathbf{j}_1-\mathbf{j}_2+\mathbf{j}_3) = \sqrt{⅓}\,(-\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3).
\end{aligned}$$

See Figure 6.23.

In the same way, the regular tetrahedron basis can be made from the sinistral combination, etc.

In principle, the regular tetraon four unit 1-vector basis $\{\mathbf{u}_\mu\}$ has no link to any specific Cartesian basis. It is free as long as it fulfils (6.460). From this we presume an autonomous regular tetraon four-angular-momenta function (6.464) $\Psi^{pag\text{-}2}_{\text{auto},\wedge} = \sum_\mu \mathbf{S}_\mu$. This will stand in a complementing dialectic relation with an angular development versor wavefunction as (6.451). Such versor function, of course, depend on quaternion basis $\{1, \boldsymbol{i}\boldsymbol{\sigma}_1, \boldsymbol{i}\boldsymbol{\sigma}_2, \boldsymbol{i}\boldsymbol{\sigma}_3\}$ that in our idea is built on a Cartesian basis. But we presume it is free in the lab frame and autonomous for just one ***entity*** $\Psi_{½}$.

6.5.10.7. Multiple Circular Oscillators as Structure Form *Qualities* for Spin½ Fermions

It may seem strange that we dare to suggest breaking up an indivisible ***entity*** $\Psi_{½}$ into four oscillators. We first recall the fundamental dialectic complement between angular momentum and cyclic oscillation. The constant angular development appoints area ***direction***[379] in 3-space. We count four area ***direction*** components with half-reduced magnitude amplitude of the angular momenta. The development of these exists in circular oscillators, as we just above § 6.5.10.2 have chosen as four angular area oscillators that we may describe by 1-spinor wavefunctions like (6.166) as

(6.486) $$\psi^{½}_{\mu\pm} \sim \varrho_\mu e^{\pm½\boldsymbol{i}\mathbf{u}_\mu\varphi_\mu} = \varrho_\mu\left(\cos ½\varphi_\mu \pm \boldsymbol{i}\mathbf{u}_\mu \sin ½\varphi_\mu\right),$$

with their angular momenta $\mathbf{S}_\mu = \pm\tfrac{1}{2}\boldsymbol{i}\mathbf{u}_\mu$ in the symmetric ***directions*** of a regular tetraon (6.460)

[379] We also recall that locality of a fermion $\Psi_{½}$ in 3 space is a consequence of the intersection of at least three planes. In the Platonic symmetry, this is four plane faces of a regular tetrahedron with a circumscribed center.

positively orientated ***outwards***[380] resulting in an impact, that is ***direction*** indifferent expressed as fractional scalars ***quantities*** (6.474) $q = 1, \tfrac{2}{3}, \tfrac{1}{3}, 0, -\tfrac{1}{3}, -\tfrac{2}{3}, -1$, as a result of covariance.
Inside one ***entity*** $\Psi_{½}$, the circular oscillator functions are combined in one spinning function

(6.487) $$\Psi^{\text{spinning}}_{\text{auto},\wedge} = \textstyle\sum_{\mu=0}^{3}\varrho_\mu\cos½\varphi_\mu + \sum_{\mu=0}^{3}\boldsymbol{i}\mathbf{u}_\mu\varrho_\mu\sin½\varphi_\mu = u_0 + \mathrm{B} = \langle\Psi^{\text{spinning}}_{\text{auto},\wedge}\rangle_0 + \langle\Psi^{\text{spinning}}_{\text{auto},\wedge}\rangle_2.$$

The autonomy condition for the indivisible $\Psi_{½}$ is that the four symmetric functions (6.486) are chronometric synchronised as $\varphi_\mu = \omega t_\varphi + \theta_\mu$. The autonomy chronometer is $|\omega|=1 \Rightarrow \varphi = t_\varphi$.[381]
We presume that it is possible (for the autonomous $\Psi_{½}$)[382] to find the dilation ϱ_μ and phase angle φ_μ parameters to fit as (6.431) combined in (6.432) reaching (6.433) and (6.434) giving the versor (6.435)⟵(6.145) that is founded on the orthonormal quaternion basis of the $\mathbb{H}\sim\mathcal{G}_{0,2}(\mathbb{R})$ algebra.

We have used three different pictures with four, three, and down to two circular oscillators with belonging angular momentum ***direction*** component ***qualities*** to describe the ***primary qualities*** of one indivisible fermion ***entity*** $\Psi_{½}$ as a 3-space locality (intersection of three independent planes).
To prevent the whole story of these going totally mysterious we recall the full geometric algebra of $\mathcal{G}_3(\mathbb{R})$ with the upper ***grade*** limit $\mathbf{a}\wedge\boldsymbol{i} = 0$ and repeat that $\mathcal{G}_{0,2}(\mathbb{R})$ is a closed even algebra of this $\mathbb{H}\sim\mathcal{G}_{0,2}(\mathbb{R}) \subset \mathcal{G}_3(\mathbb{R})$. The standard basis (6.126)⟶(6.134) $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3 := \boldsymbol{i}_1\boldsymbol{i}_2\}$ express from definition (6.123) the fundamental *interconnectivity* of the bivectors inside $\mathcal{G}_{0,2}(\mathbb{R})$.
This *interconnectivity* idea is inherited directly to the angular momentum bivectors as (6.291a)

(6.488) $$[\mathbf{L}_1, \mathbf{L}_2] = \mathbf{L}_1\mathbf{L}_2 = \hbar\tfrac{1}{2}\mathbf{L}_3 \quad\Longrightarrow\quad \tfrac{2}{\hbar}\mathbf{L}_3 = \tfrac{2}{\hbar}\mathbf{L}_1\tfrac{2}{\hbar}\mathbf{L}_2 \quad\Longleftarrow\quad \boldsymbol{i}_3 := \boldsymbol{i}_1\boldsymbol{i}_2.$$

This is the a priori foundation idea of *spin one-half* (spin½) for any indivisible fermionic ***entity*** $\Psi_{½}$ as a locality in 3-space of physics, represented by a full geometric algebra $\mathcal{G}_3(\mathbb{R})$.
Hamilton's original idea from 1843 of versor quaternion basis $\{1,\ \boldsymbol{i} = \boldsymbol{i}_1,\ \boldsymbol{j} = \boldsymbol{i}_2,\ \boldsymbol{k} = \boldsymbol{i}_3\}$ where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$, make the fundamental internal structure of a fermion $\boldsymbol{k} = \boldsymbol{ij}$.
This structure is given inside the even versors in the unitary geometric algebra of $\mathcal{G}_{0,2}(\mathbb{R})$.
What Hamilton did not realise, is that these bivectors have to be created from an underlying 1-vector concept $\boldsymbol{i} = \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ (5.72) etc. (6.31) of a more comprehensive algebra $\mathcal{G}_3(\mathbb{R})$.

6.5.11. The Full Geometric Algebra $\mathcal{G}_3(\mathbb{R})$ for Spin½ Fermions

We have above seen that we cannot make an external measurable observable out of the sum of the three bivector components $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3$ (6.316) or the dual 1-vector form (6.317) for $\mathbf{j}$.
Instead, we use the orthogonal Pythagorean square sum (6.322) $J^2 = J_1^2 + J_2^2 + J_3^2 = \hbar^2\lambda_k\lambda_k \geq 0$, as a commutating Casimir operator that we separate into two parts

(6.489) $$J^2 = J_\perp^2 + J_3^2, \quad \text{where } J_\perp^2 = J_1^2 + J_2^2 = \hbar^2(\lambda_1^2+\lambda_2^2), \text{ using } J_k^2 = \tfrac{1}{2}\left(\mathbf{j}_k^2 - \mathbf{L}_k^2\right).$$

6.5.11.2. The Non Quaternion *Grades* ≤3 for of Indivisible *Entities* $\Psi_{½}$ in 3 Space

The ladder operators defined in (6.306)-(6.307) J_+ and J_- involve ***direction*** 0< ***grades*** ≤3.
In (6.367)-(6.368) they are used to change the state of 3-space for fermion ***entities*** $\Psi_{½}$:

(6.490) $$J_\pm\left|\tfrac{3}{4}, \mp\tfrac{1}{2}\right\rangle \doteq \hbar 1\left|\tfrac{3}{4}, \pm\tfrac{1}{2}\right\rangle,$$

Chance the spin quantum number from $-½\hbar$ to $+½\hbar$ and vice versa by *one whole* $\pm1\hbar$ for $\Psi_{½}$.
The two ladders J_+ and J_- do not *commute* and by that the operator construction (6.315)

(6.491) $$J_+J_- - J_-J_+ = 2\hbar\mathbf{j}_3 = -\boldsymbol{i}2\hbar\mathbf{L}_3$$

[380] We presume (6.486) to be covariant with their outwards ***directions*** $\mathbf{u}_\mu$ due to the orthogonality to the dual transversal $\boldsymbol{i}\mathbf{u}_\mu$, in that, we consider the contravariant definition of the regular tetraon symmetry $\mathbf{u}_\mu\cdot\mathbf{u}_\nu = -⅓$, for $\mu\neq\nu$ in (6.460).
[381] In an external laboratory environment, we can estimate the internal 'pitch' frequency energy by $\omega_{½} \sim ½mc^2/\hbar$. Then the four phases are $\varphi_\mu = \omega_{½}t + \theta_\mu$, $\forall\theta_\mu \in \odot_\mu$. We could call these oscillators for *The Playing Music Quartet* of the indivisible ***entity*** $\Psi_{½}$.
[382] Warning, do not dare to do this, think of William of Ockham (non nesesante) ⊙. Perhaps it could lead to some Hopf fibration.