(6.404) $\quad \psi^{\frac12\odot}_{k\pm} \sim |\pm\tfrac12\rangle_k \sim \quad \tilde r(\rho)\odot_k e^{\pm \boldsymbol{i}_k \frac12 \phi_k} \quad \sim \quad$ (simplified as) $\quad \widehat{\psi^{\frac12}_{k\pm}} \sim e^{\pm \boldsymbol{i}_k \frac12 \phi_k} \quad \in \mathbb{H}.$ [358]

Each with specified $\boldsymbol{i}_k$ ***directions***, and all three acting together simultaneously.

These are endowed with angular momentum eigenvalue bivectors $\mathbf{L}^{\pm}_k = \pm\tfrac12\hbar\boldsymbol{i}_k = \pm\tfrac12\hbar\boldsymbol{i}\boldsymbol{\sigma}_k$

The equivalent rotors $U = e^{\frac12 \boldsymbol{i}\mathbf{b}} \sim e^{\boldsymbol{i}_k \frac12 \phi_k}$ possess the transversal plane normal-axial cylinder symmetry around the 1-vector $\mathbf{b}$. This we know from the regular rotation (6.70) that acts on an arbitrary geometric (multi)-vector $\mathbf{x}$ by canonical sandwiching by the rotor and its reversed

(6.405) $\quad \mathcal{R}\mathbf{x} = U\mathbf{x}U^\dagger = e^{\frac12 \boldsymbol{i}\mathbf{b}}\mathbf{x}\, e^{-\frac12 \boldsymbol{i}\mathbf{b}} \quad = e^{\boldsymbol{i}\mathbf{b}}\mathbf{x}$

Whence we instead of the subton stats (6.402) in these ***directions*** use the rotor stats (6.404)

(6.406) $\quad U = e^{+\frac12 \boldsymbol{i}\mathbf{b}} \rightarrow \qquad \widehat{\psi^{\frac12}_{k\pm}} = \quad e^{\pm \boldsymbol{i}_k \frac12 \phi_k},$

as templet proportional to the spinor states for each $k = 1{,}2{,}3$, with the symmetries $\odot_k = e^{\boldsymbol{i}_k\theta}$

(6.407) $\quad \psi^{\frac12}_{k\pm}(\rho) = \tilde r(\rho)\odot_k e^{\pm \boldsymbol{i}_k \frac12 \phi_k}$, for $\forall \rho \ge 0$, $\quad$ where $\int_0^\infty \psi^{\frac12}_{k\pm}\psi^{\dagger\frac12}_{k\pm} d\rho = \langle \psi^{\frac12}_{k\pm} | \psi^{\frac12}_{k\pm}\rangle = \frac14$.

We have three such components of orthogonal oscillating spinor states endowing angular momenta. We remember that these states do not commute as described in § 6.3.5.2, therefor their individual independent ***directions*** $\boldsymbol{\sigma}_k$ cannot be reduced. (Their interconnectivity does the impact.) They exist in the planes of the exponential functions $e^{\pm \boldsymbol{i}_k \phi_k}$, that is the plane supported by the unit argument ***direction*** indicated by $\boldsymbol{i}_k$, which is the free eigen-planes of $\mathbf{L}^{\pm}_k = \pm\hbar\boldsymbol{i}_k = \pm\hbar\boldsymbol{i}\boldsymbol{\sigma}_k$.

For simplicity, we remove the distribution factor $\tilde r(\rho)\odot_k$ and replace it with the amplitude ϱ_k

(6.408) $\quad \psi^{\frac12}_{k\pm} = \varrho_k e^{\pm \boldsymbol{i}_k \frac12 \phi_k} \quad = \varrho_k(\cos\tfrac12\phi_k \pm \boldsymbol{i}_k \sin\tfrac12\phi_k) \quad \in \mathbb{H},$ $\qquad$ as (6.164)-(6.166) .

These three $\mathcal{G}_{0,2}(\mathbb{R})$ multivector state components can be linear combined for a full ***entity*** $\Psi_{\frac12}$

(6.409) $\quad \psi^{\frac12}_{\pm} = \psi^{\frac12}_{3\pm} + \psi^{\frac12}_{2\pm} + \psi^{\frac12}_{1\pm} \quad \in \mathbb{H}.$

Special for the external chosen ***direction*** $\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3$, we change $\varrho = \varrho_3$ and $\phi = \phi_3$ (see (6.146))

(6.410) $\quad \psi^{\frac12}_{3+} = \varrho_3 e^{+\boldsymbol{i}_3 \frac12 \phi_3} \quad = \varrho(\cos\tfrac12\phi + \boldsymbol{i}_3 \sin\tfrac12\phi) \quad \in \mathbb{H}.$

For the other ***directions***, we left multiply operate this by a quaternion spinor factor $\frac12\frac{\rho}{\varrho}\boldsymbol{i}_2 e^{\boldsymbol{i}_3 \frac12\theta}$

(6.411) $\quad \frac{\rho}{2\varrho}\boldsymbol{i}_2\odot_3\psi^{\frac12}_{3+} \sim \quad \psi^{\frac12}_{2+} = \tfrac12\rho(\boldsymbol{i}_2\cos\tfrac12\psi + \boldsymbol{i}_1\sin\tfrac12\psi), \quad$ with $\psi = +\phi + \theta$.

or operate (6.410) by the factor $\pm\frac{\rho}{2\varrho}\boldsymbol{i}_1 e^{\boldsymbol{i}_3 \frac12(\theta\mp\pi)}$ and using e.g., $\pm\psi_1 = \psi \mp \pi \leftarrow \phi + \theta$.

(6.412) $\quad \pm\frac{\rho}{2\varrho}\boldsymbol{i}_1\odot_3\psi^{\frac12}_{3+} \sim \quad \psi^{\frac12}_{1\pm} = \pm\tfrac12\rho(\boldsymbol{i}_1\cos\tfrac12\psi_1 - \boldsymbol{i}_2\sin\tfrac12\psi_1) \;=\; \tfrac12\rho(\boldsymbol{i}_1\sin\tfrac12\psi + \boldsymbol{i}_2\cos\tfrac12\psi).$

Here we now use ρ as a real amplitude factor.[359] We note the phase orientation for $\pm\frac12\boldsymbol{i}_1$.

These two (6.411)-0 are the same, whereby we can remove the factor $\frac12$ and get

(6.413) $\quad \psi^{\frac12}_{1+} = \quad \psi^{\frac12}_{2+} + \psi^{\frac12}_{1+} = \quad \rho(\boldsymbol{i}_2\cos\tfrac12\psi + \boldsymbol{i}_1\sin\tfrac12\psi) \qquad = \rho\boldsymbol{i}_\perp(\tfrac12\psi),$

as a spinning bivector oscillation shown in Figure 6.16 in agreement with the impact of the interconnectivity expressed in (6.151)-(6.155). The independency $\phi \ne \psi$ is a phase factor θ. We note a ***quality***, a phase shift 2π in e.g. ψ_1 change the orientation of that contribution $\psi^{\frac12}_{1\pm}$.

6.5.8.3. The Oscillator Fluctuating Versor Wavefunction for the *Entity* $\Psi_{\frac12}$ in 𝟛 space

The superposition of these by (6.409) gives the quaternion (see § 6.4.4.3)

(6.414) $\quad \psi_{\pm\frac12} = \psi^{\frac12}_{3\pm} + \psi^{\frac12}_{2\pm} + \psi^{\frac12}_{1\pm} = \quad \varrho(\cos\tfrac12\phi + \boldsymbol{i}_3\sin\tfrac12\phi) + \rho(\boldsymbol{i}_2\cos\tfrac12\psi + \boldsymbol{i}_1\sin\tfrac12\psi)$

$\qquad\qquad \sim \; U = \hat Q = \; u_0 + u_3\boldsymbol{i}_3 + u_1\boldsymbol{i}_1 + u_2\boldsymbol{i}_2 \qquad \in \mathbb{H} \sim \mathcal{G}_{0,2}(\mathbb{R}).$

This we compare to (6.168) using the definitions (6.145f)-(6.150) and find that the versor quaternion is fully capable to describe an ***entity*** $\Psi_{\frac12}$ in 𝟛-space.

[358] The double k index $\boldsymbol{i}_k\phi_k$ in the exponent is not summed here in this § when we are looking at the components. This is due to the causal sequential product operations do not commute.

[359] Here not as a polar coordinate for density distribution $\tilde r(\rho)$ in physical space. The replacement is rather $\rho \leftrightarrow \langle \tilde r(\rho)\rangle$.

This quaternion 2-rotor is unitary if (6.150) is fulfilled $\varrho^2 + \rho^2 = u_0^2 + u_3^2 + u_1^2 + u_2^2 = 1$

We also see that a free oscillating versor quaternion is governed by two angular development parameters, e.g. given as (6.148)-(6.149) performing 1-spinor oscillations (6.146)-(6.147)

(6.415) $\quad Q_3 = (u_0 + u_3\boldsymbol{i}_3) = \varrho\, e^{+\boldsymbol{i}_3\frac12\phi} \;\in \mathbb{H} \quad \leftrightarrow \quad \alpha = z_3 = \varrho e^{+i\frac12\phi} \;\in \mathbb{C},$ $\qquad$ as (6.170),

(6.416) $\quad Q_1 = (u_2 - u_1\boldsymbol{i}_3) = \rho\, e^{-\boldsymbol{i}_3\frac12\psi} \;\in \mathbb{H} \quad \leftrightarrow \quad \beta = z_1 = \rho e^{-i\frac12\psi} \;\in \mathbb{C},$ $\qquad$ as (6.171).

The condition (6.150) fulfil a 4-dimensional $\mathbb{R}^4$ unit sphere with S^3 symmetry

(6.417) $\quad S^3 = \{\forall(u_0,u_1,u_2,u_3) \in \mathbb{R}^4 \mid u_0^2+u_1^2+u_2^2+u_3^2 = 1\} = \{\forall(z_1,z_3) \in \mathbb{C}^2 \mid |z_1|^2 + |z_3|^2 = 1\} = \{\forall \hat Q \in \mathbb{H} \mid |\hat Q| = 1\}.$

This abstract S^3-*spherical symmetry* of 𝟛-space is the fundament for the isomorphic structure of the versor-quaternion group of the geometric algebra, the lifted Pauli group, and the 2×2 complex matrix group $SU(2)$ (6.175) in a way that its elements possess the ***primary quality*** of a *free **entity*** $\Psi_{\frac12}$ in 𝟛-space.

All the possible traditional two angular *spherical coordinates* $(1,\theta,\varphi)$, the polar angle θ, and the azimuthal angle φ describe all ***directions*** in a unit sphere. For completeness, we include a third quantum mechanics phase angle from $\odot \leftrightarrow U(1)$ for the overall symmetry consideration.

The total spherical symmetry is broken by an external field possessing angular momentum creating a conical precession of angular momentum of an ***entity*** $\Psi_{\frac12}$ interpreted from the external as displayed in Figure 6.22. This field can be established by an inhomogeneous magnetic field as in the Stern-Gerlach experiment. The field gradient with a lab frame ***direction*** $\boldsymbol{\sigma}_3 = \mathbf{e}_3$ consist of free subtons $\psi_{3\pm} \sim e^{\pm\boldsymbol{i}_3\omega_s t}$ (6.402)[360], that possess angular momentum $\mathbf{L}^{\pm}_3 = \pm\hbar\boldsymbol{i}_3 = \pm\hbar\boldsymbol{i}\boldsymbol{\sigma}_3$, with magnitude ***quantum*** $\hbar 1$, that interact as the symmetry braking mechanism.

Here the exchange ***quantum*** is the subton Ψ_{ω_s} frequency energy $\hbar\omega_s$ as kinetic energy with the line momentum $(\hbar/c)\omega_s$ of one subton delivered *to* or *from* each ***entity*** $\Psi_{\frac12}$.

The amount $\hbar\omega_s$ *absorbed in* or *emitted from* $\Psi_{\frac12}$ by the subton is reasonably small compared to the internal oscillation energy (“mass”) of $\Psi_{\frac12}$, making it move *up* or *down* along the gradient[361] $b(x_3)\mathbf{e}_3$ in the $\mathbf{e}_3$ ***direction*** of the inhomogeneous magnetic field as a transversal plane bivector $b(x_3)\boldsymbol{i}\mathbf{e}_3$ in the experiment. Then the total angular momentum of $\Psi_{\frac12}$ must be aligned to do a precession around $\boldsymbol{\sigma}_3 = \mathbf{e}_3$ along $\boldsymbol{i}\mathbf{e}_3$ with the projection on this $\mathbf{L}^{\Psi_{\frac12}}_3 = \pm\frac12\hbar\boldsymbol{i}_3 = \pm\frac12\hbar\boldsymbol{i}\mathbf{e}_3$.

The thought is that the first interaction with each $\Psi_{\frac12}$ aligned spin½ parallel or antiparallel to the gradient randomly from the prerequisite ***direction*** of $\Psi_{\frac12}$. Then the gradient ***direction*** locks the orientation of $\mathbf{L}^{\Psi_{\frac12}}_3$ to $+\frac12\hbar\boldsymbol{i}\mathbf{e}_3$ or $-\frac12\hbar\boldsymbol{i}\mathbf{e}_3$. Further interaction with the magnetic field is then a polarized acceleration.

The S^3-spherical symmetry is broken to an intuitive conical lab average shown in Figure 6.22. The reader should compare this to (6.218)-(6.219), (6.223) considering the autonomous magnitude factor $|\mathbf{j}^{\Psi_3}_{\pm}| = \sqrt{3/4}|\mathbf{n}|$ and note (6.140)

(6.418) $\quad \mathbf{n} := (u_1\boldsymbol{\sigma}_1 + u_2\boldsymbol{\sigma}_2 + u_3\boldsymbol{\sigma}_3)/\sqrt{1-u_0^2} \;=\; n_1\boldsymbol{\sigma}_1 + n_2\boldsymbol{\sigma}_2 + n_3\boldsymbol{\sigma}_3,$ where $|\mathbf{n}| = 1$,

and compare with the S^2 *spherical symmetry*

(6.419) $\quad S^2 = \{\forall(n_1,n_2,n_3) \in \mathbb{R}^3 \mid n_1^2 + n_2^2 + n_3^2 = 1\},$

where the ***direction*** symmetry is broken by the projection (6.379), (6.388) $\;\mathbf{j}_3 \rightarrow \mathbf{j}^{\Psi_{\frac12}}_3 = \pm\frac12\hbar\boldsymbol{\sigma}_3$, that we interpreted from the external frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\boldsymbol{\sigma}_3 = \mathbf{e}_3$, therefor we write

(6.420) $\quad \mathbf{j}^{\Psi_{\frac12}}_3 = \pm\frac12\hbar\mathbf{e}_3.$ [362]

[360] The thought experiment here is, that the static magnetic gradient consists of up subtons $e^{+\boldsymbol{i}_3\omega_s t}$ and down subtons $e^{-\boldsymbol{i}_3\omega_s t}$, which as energy flow balance each other but give a resulting angular momentum, that can interact with our ***entity*** $\Psi_{\frac12}$.

[361] We use a real scalar function $b(x_3) \in \mathbb{R}$ as a coefficient to the ***direction*** $\mathbf{e}_3$ to describe the gradient field along the coordinate x_3.

[362] Remember that the unit of $\mathbf{e}_3$ must be one $|\mathbf{e}_3| = 1 = \hbar$ in a ***quantum*** unit system. In other systems, $\mathbf{e}_3$ can have the unit $[\hbar^{-1}]$.