Vice versa, seen from the lab the total angular momentum 1-vector $\mathrm{j}_{\pm}^{\Psi_{1/2}} = -\boldsymbol{i}\mathrm{L}_{\pm}^{\Psi_{1/2}}$ of $\Psi_{1/2}$ is making a conical precession displayed in Figure 6.22.
This lab relation oscillation does not affect the chosen ***direction*** $\sigma_3 = \mathbf{e}_3$.

6.5.7. Synthesis of the Locality of *Entities* in 𝔍-space

In this § we omit $\hbar=1$ display in the formulas to keep the argumentation of the idea loud and clear.
We have above argued for the three angular momentum component possibilities (as operators) $\mathrm{L}_1, \mathrm{L}_2$ and L_3 in a 𝔍-space interpreted as bivectors following the geometric algebra $\mathbb{H}\sim \mathcal{G}_{0,2}(\mathbb{R})$ expressed in the rules (6.123) and (6.130) as $\boldsymbol{i}_1 = \boldsymbol{i}_2\boldsymbol{i}_3$, $\boldsymbol{i}_2 = \boldsymbol{i}_3\boldsymbol{i}_1$, $\boldsymbol{i}_3 = \boldsymbol{i}_1\boldsymbol{i}_2$ and this we write out by the general multivector product rule that handles the antisymmetries (6.281)⟵(6.55)

(6.393) $$\boldsymbol{i}_1 = \tfrac{1}{2}(\boldsymbol{i}_2\boldsymbol{i}_3 - \boldsymbol{i}_3\boldsymbol{i}_2), \qquad \boldsymbol{i}_2 = \tfrac{1}{2}(\boldsymbol{i}_3\boldsymbol{i}_1 - \boldsymbol{i}_1\boldsymbol{i}_3), \qquad \boldsymbol{i}_3 = \tfrac{1}{2}(\boldsymbol{i}_1\boldsymbol{i}_2 - \boldsymbol{i}_2\boldsymbol{i}_1).$$

This expresses the interconnected antisymmetries between the quaternion perpendicular basis planes. The quantum mechanical antisymmetric commutation relation is expressed in (6.291)-(6.293). While the corresponding geometric products are simplified in (6.291a)-(6.293a).
All these antisymmetric interconnectivity relation shapes (gestalten) an entanglement of angular momentum subjects we now call bivector-eigenvalues for the physical subject planes (see (6.299))

(6.394) $$\mathrm{L}_1 = \tfrac{1}{2}\boldsymbol{i}_1, \qquad \mathrm{L}_2 = \tfrac{1}{2}\boldsymbol{i}_2, \qquad \mathrm{L}_3 = \tfrac{1}{2}\boldsymbol{i}_3, \qquad \text{or} \qquad \mathrm{L}_k = \tfrac{1}{2}\boldsymbol{i}_k,$$ [355]

that automatically fulfils the basic angular momentum ***quanta*** (6.291a)-(6.293a)

(6.395) $$\mathrm{L}_2\mathrm{L}_3 = \tfrac{1}{2}\mathrm{L}_1, \qquad \mathrm{L}_3\mathrm{L}_1 = \tfrac{1}{2}\mathrm{L}_2, \qquad \mathrm{L}_1\mathrm{L}_2 = \tfrac{1}{2}\mathrm{L}_3.$$

Two perpendicular plane objects, e.g., those represented by $\mathrm{L}_1 = \tfrac{1}{2}\sigma_3\sigma_2$ and $\mathrm{L}_2 = \tfrac{1}{2}\sigma_1\sigma_3$ intersect in a geometric line represented by the ***direction*** 1-vector object σ_3. This line intersects its own dual transversal plane object represented by $\mathrm{L}_3 = \tfrac{1}{2}\boldsymbol{i}_3 = \tfrac{1}{2}\sigma_2\sigma_1$ just in one point.
Three eigen-bivector angular momentum *perpendicular* plane objects (algebraically *orthogonal*) always intersect in one point and just one point in 𝔍-space! Therefore, one fundamental physical ***entity*** Ψ_3 formed by these three angular momentum planes possess that property of locality as the ***quality*** idea Leibniz called *analysis situs*.
I now call it *locus situs* which makes the intersection of these three planes to the center point of an ***entity*** Ψ_3 in physical 𝔍-space.
Contrary we have the traditional dual picture with three 1-vector eigenvalue angular momenta

(6.396) $$\mathrm{j}_1 = \tfrac{1}{2}\sigma_1, \qquad \mathrm{j}_2 = \tfrac{1}{2}\sigma_2, \qquad \mathrm{j}_3 = \tfrac{1}{2}\sigma_3, \qquad \text{or} \qquad \mathrm{j}_k = \tfrac{1}{2}\sigma_k.$$

We can intuit these generator ***directions*** as geometric linear axis, that can perform free line translations in 𝔍-space, so that they likely do not intersect. The tradition has talked in a religious way to choose a point as origo for these three axes to intersect in a cartesian coordinate system.
Fortunately, we are saved from this religion by the fact that (6.396) as (6.395) automatically fulfils (6.291a)-(6.293a) as three transversal bivectors formed by a geometric product of orthogonal pairs of 1-vectors, that gives the concept of three transversal planes that always intersects

(6.397) $$\mathrm{j}_3\mathrm{j}_2 = \boldsymbol{i}\tfrac{1}{2}\mathrm{j}_1, \qquad \mathrm{j}_1\mathrm{j}_3 = \boldsymbol{i}\tfrac{1}{2}\mathrm{j}_2, \qquad \mathrm{j}_2\mathrm{j}_1 = \boldsymbol{i}\tfrac{1}{2}\mathrm{j}_3,$$ these are equivalent to those in (6.395).

These products send the 1-vectors into the closed even quaternion (lifted Pauli) algebra.
In all, the idea of quaternions basis in a real even closed geometric algebra $\mathbb{H}\sim \mathcal{G}_{0,2}(\mathbb{R})$ helps us by the angular plane concept to see, that they autonomously are making (gestalten) a center point as a locality of an ***entity*** Ψ_3 in physical 𝔍-space. This locality center can be the origo point for the idea of cyclic oscillations.
These oscillations are mandatory to the idea of some internal phase angular frequency energy $\hbar\omega$ and to the idea of in space local situated energy at all. The tradition calls this mass: $m = \omega\hbar/c^2$.

[355] The orientation possibilities ± is omitted because we only need the basic ***directions*** for the angular momentum basis.

6.5.8. The Idea of One Spin½ *Entity* in Physical 𝔍-space

The interesting thing with the above treatment is that there can be three orthogonal circular oscillators, each with angular momentum ***quantum*** number one-half $\tfrac{1}{2}\hbar$ with a strongly entangled interconnectivity formed by the quaternion idea of the real even geometric algebra $\mathbb{H}\sim \mathcal{G}_{0,2}(\mathbb{R})$.
These three oscillator planes always form by intersection a locality center for a physical ***entity*** $\Psi_{1/2}$.
If each of these three oscillators were free in the external, they would possess an angular momentum of one-whole $1\hbar$. Now when they are strongly bound to their intersection center the angular momentum is only one-half $\tfrac{1}{2}\hbar$ for each component.

6.5.8.1. The Extension Distribution of the Wavefunctions

To get a deeper understanding, complementary to the angular momentum ***direction*** idea, let us look back to section I. 3.3.1, where we for the circular oscillator have the excited probability density magnitude (3.143)[356]

(6.398) $$\tilde{r}(\rho) = \tfrac{1}{\sqrt[4]{\pi}}\rho e^{-\frac{1}{2}\rho^2} \in \mathbb{R}, \ \text{for} \ \forall\rho \in \mathbb{R},$$

which is an odd function of form I. (3.120) $\tilde{r}(\rho) = -\tilde{r}(-\rho)$, that in a geometric linear ***pqg***-1 way is in balance by Newton's third law and double the probability density (3.144) for $\rho \geq 0$.
This ρ is the radial plane polar coordinate in the plane idea. Now that ***entity*** $\Psi_{1/2}$ has a locality center with a spherical S^2 symmetry of 𝔍-space this balance is broken and shared with the other plane ***directions***. Therefore, only positive radial $\rho \geq 0$ make sense in the radial probability density of excitation of the eternal ground state. Of course, the total has in an integrated way to average to unity $\langle\Psi_{1/2}\rangle=1$, but $\langle\tilde{r}(\rho)\rangle = ½$, or as an associate to (3.150)

(6.399) $$\langle\tilde{r}(\rho)|\tilde{r}(\rho)\rangle = \int_0^\infty \tfrac{1}{\sqrt{\pi}}\rho^2 e^{-\rho^2} d\rho = ¼,$$

therefore, we below will omit factor 2 for radial dependency relative to the free circle oscillator

(6.400) $$\psi_{\pm}^{\odot} = |1,\pm 1\rangle = 2\tilde{r}(\rho)\odot e^{+i\omega t} \quad \in\mathbb{C},$$

as the complex scalar wavefunction of a free excited subton from (3.148)-(3.149), that is an eigenfunction to the *complex scalar* eigenvalue equation $\hat{L}|1,\pm 1\rangle \doteq \pm 1\hbar|1,\pm 1\rangle$.
Alternative to the treatment in chapter I. 3.3 of the quantum circle oscillator state $\psi_{\pm}^{\odot}$, we now can indicate the plane ***direction quality*** by $\boldsymbol{i}_k$, and write this ***direction*** component of the angular momentum operator as $\mathrm{L}_k = \lambda_k\boldsymbol{i}_k$, and the eigenvalue equation (3.114) →(6.231) as

(6.401) $$\lambda_k\boldsymbol{i}_k|\pm 1\rangle_k \doteq \pm 1\hbar\boldsymbol{i}_k|\pm 1\rangle_k.$$

This gives the free single ***directional*** eigen-bivector angular momentum $\mathrm{L}_k^{\pm} = \pm\hbar\boldsymbol{i}_k = \pm\hbar\boldsymbol{i}\sigma_k$.
The optional free subton eigen-rotor (wavefunction) for this we express as

(6.402) $$\psi_{k\pm}^{\odot} \sim |\pm 1\rangle_k \sim \ 2\tilde{r}(\rho)\odot_k e^{\pm \boldsymbol{i}_k\omega_k t} \sim \ e^{\pm\boldsymbol{i}_k\omega_k t} \in\mathbb{H}. \qquad (\text{Only one free } \omega_k=\omega_s, \omega_{j\neq k}=0).$$

These external ***direction*** components are transversal plane circle oscillators. The free subton is what performs the communication of the ***entity*** $\Psi_{1/2}$ with the external surroundings by exchange of ***quanta*** of line momentum $(\hbar/c)\omega_s$ and kinetic energy by subton frequency energy $\hbar\omega_s$.[357]

6.5.8.2. The Internal Oscillating Wavefunction Components

In the internal even $\mathcal{G}_{0,2}(\mathbb{R})$ algebra idea for these, we have the strong interconnectivity (6.123), (6.130) and (6.291a)-(6.293a) whereby we construct the component eigenvalue equations

(6.403) $$\lambda_k\boldsymbol{i}_k|\pm ½\rangle_k \doteq \pm ½\hbar\boldsymbol{i}_k|\pm ½\rangle_k.$$

The interconnectivity entangled state components of eigen-spinors (phase wavefunctions) are

[356] The dilation factor ρ is transformed to probability density factor $\tilde{r}(\rho)$ of a radial coordinate ρ from a center in a plane.
[357] We may presume that the external frequency energy ω_s of subtons is much less than an oscillation internal in the ***entity*** $\Psi_{1/2}$, that external appears as mass $m\sim(\hbar/c^2)\omega_k$. This we hide autonomous $|\omega_k|=1$, so we only concern the phase parameter. Therefore a subton exchange does not alter the integrity of ***entity*** $\Psi_{1/2}$, but only breaks the ***direction*** symmetry by the angular momentum magnitude ***quantum*** $\hbar=1$. – More below § 6.5.8.3.