bivector component in that ***direction*** $\langle \mathbf{L}_3\rangle_{½} = \mathbf{L}_3^{\Psi_{½}} = \pm\frac{1}{2}\hbar\boldsymbol{i}_3$.

This eigenstate $\pm\frac{1}{2}\hbar\boldsymbol{i}_3 \,|\, \frac{3}{4}, \pm\frac{1}{2}\rangle$ looks like the half of an external subton state ***direction***, (6.232).

The other components are perpendicular oscillating angular momentum combined as

(6.375) $\mathbf{L}_\perp = \mathbf{L}_1 + \mathbf{L}_2 = \hbar\lambda_1\boldsymbol{i}_1(\phi) + \hbar\lambda_2\boldsymbol{i}_2(\phi) = \hbar e^{\boldsymbol{i}_3\phi}\boldsymbol{i}(\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2)$ ,

with $\pm\boldsymbol{i}_3$ orientation of the oscillating ***direction*** hidden in the angular development $\phi = \pm\omega t$.

For the intuition of this, we presume $\langle\lambda_1\rangle = \langle\lambda_2\rangle = \pm½$ for[352]

(6.376) $\mathbf{L}_\perp^{\Psi_{½}} = \langle\mathbf{L}_\perp\rangle_{½} = \langle\mathbf{L}_1 + \mathbf{L}_2\rangle_{½} \approx \pm\frac{1}{2}\hbar\big(\boldsymbol{i}_1(\phi) + \boldsymbol{i}_2(\phi)\big) = \pm\frac{1}{2}\hbar e^{\boldsymbol{i}_3\phi}\boldsymbol{i}(\mathbf{e}_1 + \mathbf{e}_2) = \pm\frac{1}{2}\sqrt{2}\hbar\boldsymbol{i}_\perp(\phi)$.

The resulting total angular momentum oscillating bivector (rotating along the $\boldsymbol{i}_3$ plane)

(6.377) $\mathbf{L}_\pm^{\Psi_{½}} = \mathbf{L}_\perp^{\Psi_{½}} + \mathbf{L}_3^{\Psi_{½}} = \pm\frac{1}{2}\hbar e^{\boldsymbol{i}_3\phi}\boldsymbol{i}(\mathbf{e}_1 + \mathbf{e}_2) \pm \frac{1}{2}\hbar\boldsymbol{i}\boldsymbol{\sigma}_3 = \pm\frac{1}{2}\hbar\big(\sqrt{2}\boldsymbol{i}_\perp(\phi) + \boldsymbol{i}_3\big)$,

and dual as oscillating 1-vectors $\boldsymbol{\sigma}_\perp(\phi)$ around $\boldsymbol{\sigma}_3$ relative to $\sqrt{½}(\mathbf{e}_1 + \mathbf{e}_2)$ we have

(6.378) $\mathbf{j}_\pm^{\Psi_{½}}(\phi) = \pm\frac{1}{2}\hbar(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3) = \pm\frac{1}{2}\hbar\big(e^{\boldsymbol{i}\boldsymbol{\sigma}_3\phi}(\mathbf{e}_1 + \mathbf{e}_2) + \boldsymbol{\sigma}_3\big) = \pm\frac{1}{2}\hbar\big(\sqrt{2}\boldsymbol{\sigma}_\perp(\phi) + \boldsymbol{\sigma}_3\big)$.

Traditional this is displayed as a cone object with height ½ and radius $\sqrt{½}$ with magnitude $\big|\mathbf{j}_\pm^{\Psi_3}\big| = \sqrt{¾}$ in Figure 6.22.[353]

This is a simulated intuition of the ***entity*** $\Psi_{½}$ spinning around the 1-vector angular momentum projection[354]

(6.379) $\mathbf{j}_3 \longrightarrow \mathbf{j}_3^{\Psi_{½}} = \pm\frac{1}{2}\hbar\boldsymbol{\sigma}_3$

in a conical precession along the plane supported by $\boldsymbol{i}_3$.

This intuition is overruled by the spherical symmetry locality in $\mathfrak{Z}$-space $\{\theta \to e^{\boldsymbol{i}\theta} \,|\, \text{for } \forall\theta \in \mathbb{R}, \text{ and } \forall\boldsymbol{i} \in \mathfrak{Z}\}$.

This symmetry demands, that all three angular momenta expressed in the interconnectivity relations (6.291)-(6.293a) equally important, but the commutation relation makes only one ***direction*** be external governing at the time, (our choice was $\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3$). We, therefore, write (6.374)

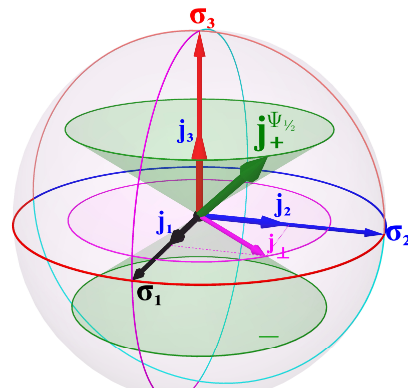


Figure 6.22

The spin½ cone with the height projection $\mathbf{j}_3^{\Psi_{½}} = \pm\frac{1}{2}\hbar\boldsymbol{\sigma}_3$ , $(\hbar = 1)$, and the transversal plane projection $\mathbf{j}_\perp(\phi) = \frac{1}{2}\hbar(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) = \frac{1}{2}\hbar\sqrt{2}\boldsymbol{\sigma}_\perp(\phi)$, resulting in a 1-vector $\mathbf{j}_\pm^{\Psi_z}(\phi)$ in a rotation symmetry $\{\theta \to e^{\boldsymbol{i}_3\theta} \,|\, \forall\theta \in \mathbb{R}\}$ representing all conical directions of an oscillating rotation $e^{\boldsymbol{i}\boldsymbol{\sigma}_3\phi}\,\mathbf{j}_\pm^{\Psi_3}(\phi + \theta)$ simultaneously $(+ \text{ or } -)$. E.g., as displayed $\mathbf{j}_\pm^{\Psi_3} = \frac{1}{2}\boldsymbol{\sigma}_1 + \frac{1}{2}\boldsymbol{\sigma}_2 \pm \frac{1}{2}\boldsymbol{\sigma}_3$. Seen from the external lab this hole frame $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ is rotating oscillating by the 1-rotor $U_\phi \equiv e^{½\boldsymbol{i}_3\phi} = e^{½\boldsymbol{i}_3(\pm\omega t + \theta)}$. By that the conical precession of the total angular momentum 1-vector.

(6.380) $\langle\lambda_r^2\rangle_{½} = \langle\lambda_1^2\rangle_{½} + \langle\lambda_2^2\rangle_{½} + \langle\lambda_3^2\rangle_{½} = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4}$.

This total external ***direction***-free symmetry suggests that

(6.381) $\sqrt{\langle\lambda_1^2\rangle_{½}} = \sqrt{\langle\lambda_2^2\rangle_{½}} = \sqrt{\langle\lambda_3^2\rangle_{½}} = \frac{1}{2}$,

from this, we deduct the mean scalar values $\langle\lambda_k\rangle_{½} = \pm\frac{1}{2}$.

In this autonomous freedom, we are given the full angular momentum operator (6.317) for the hole ***entity*** $\Psi_{½}$,

(6.382) $\mathbf{j}^{\Psi_{½}} = \hbar(\lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2 + \lambda_3\boldsymbol{\sigma}_3)$,

and dual from (6.316) the full transversal bivector operator

(6.383) $\mathbf{L}^{\Psi_{½}} = \hbar(\lambda_1\boldsymbol{i}_1 + \lambda_2\boldsymbol{i}_2 + \lambda_3\boldsymbol{i}_3)$,

for the total angular momentum. Here in this chapter 6.5 we are concerned about this total angular momentum for only one fundamental ***entity*** $\Psi_{½}$.

We have separated this in three real linear independent orthogonal 1-vector ***directions*** $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ that results in their dual transversal bivector planes supported from the quaternion basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ forming the even geometric algebra $\mathbb{H} \sim \mathcal{G}_{0,2}(\mathbb{R})$ for a 2-rotor. This makes it possible to make an epistemological new narrative for the support for the angular momenta of 1-spinor

[352] Factor $\sqrt{2}$ come from $\sqrt{\langle\mathbf{L}_1^2 + \mathbf{L}_2^2\rangle} = \sqrt{\langle\lambda_1^2\boldsymbol{\sigma}_1^2 + \lambda_2^2\boldsymbol{\sigma}_2^2\rangle} = \sqrt{\langle\lambda_1^2 + \lambda_2^2\rangle} = \sqrt{½} = \frac{1}{2}\sqrt{2} = \frac{1}{2}\sqrt{\boldsymbol{\sigma}_1^2 + \boldsymbol{\sigma}_2^2}$.

[353] The reader can compare this figuration with the orbital angular momentum cones from literature e.g., [9] figure 11.2.

[354] The reader shall note the fundamental symmetry concept of orthogonality makes the covariant projection part of the components $\mathbf{L}_\perp, \mathbf{L}_1, \mathbf{L}_2$ disappear and the projection of $\mathbf{L}_\pm^{\Psi_{½}}$ is obvious just $\mathbf{L}_3^{\Psi_{½}}$. Similar for the duals $P_{\boldsymbol{\sigma}_3}\mathbf{j}_\pm^{\Psi_{½}} \equiv \boldsymbol{\sigma}_3^{-1}\boldsymbol{\sigma}_3\cdot\mathbf{j}_\pm^{\Psi_{½}} = \mathbf{j}_3^{\Psi_{½}} = \pm\frac{1}{2}\hbar\boldsymbol{\sigma}_3$, (5.184).









oscillators in these three perpendicular planes supported by $\boldsymbol{i}_k$'s, which is *mutually orthogonal*. They are *interconnected*, first expressed from the definition in (6.123) and consolidated in § 6.5.2 as the commutator relations for the angular momentum operator ***directions***. The internal symmetry for the autonomy ***entity*** $\Psi_{½}$ makes the scalar operator components fixed as (6.300) to

(6.384) $\lambda_1 = \pm\frac{1}{2}$, $\lambda_2 = \pm\frac{1}{2}$, $\lambda_3 = \pm\frac{1}{2}$.

For these internal autonomy expectations, we prescribe the eigenstate 1-vector

(6.385) $\mathbf{j}_{\text{auto}\pm}^{\Psi_{½}} = \hbar\left(\pm\frac{1}{2}\boldsymbol{\sigma}_1 \pm \frac{1}{2}\boldsymbol{\sigma}_2 \pm \frac{1}{2}\boldsymbol{\sigma}_3\right) \quad \leftarrow \quad \langle\mathbf{j}^{\Psi_{½}}\rangle_{½} \sim \langle\mathbf{j}_1^{\Psi_{½}}\rangle_{½} + \langle\mathbf{j}_2^{\Psi_{½}}\rangle_{½} + \langle\mathbf{j}_3^{\Psi_{½}}\rangle_{½}$,

we presume the autonomous eigenstate bivector

(6.386) $\mathbf{L}_{\text{auto}\pm}^{\Psi_{½}} = \hbar\left(\pm\frac{1}{2}\boldsymbol{i}_1 \pm \frac{1}{2}\boldsymbol{i}_2 \pm \frac{1}{2}\boldsymbol{i}_3\right) \quad \leftarrow \quad \langle\boldsymbol{i}\mathbf{j}^{\Psi_{½}}\rangle_{½} = \langle\mathbf{L}^{\Psi_{½}}\rangle_{½} \sim \langle\mathbf{L}_1^{\Psi_{½}}\rangle_{½} + \langle\mathbf{L}_3^{\Psi_{½}}\rangle_{½} + \langle\mathbf{L}_3^{\Psi_{½}}\rangle_{½}$.

The full angular momentum has the magnitude

(6.387) $\left|\mathbf{j}_{\text{auto}\pm}^{\Psi_{½}}\right| = \left|\mathbf{L}_{\text{auto}\pm}^{\Psi_{½}}\right| = \hbar\frac{\sqrt{3}}{2} \quad = \left|\langle\mathbf{j}^{\Psi_{½}}\rangle\right| = \left|\langle\mathbf{L}^{\Psi_{½}}\rangle\right|$ .

The projection of $\langle\mathbf{j}^{\Psi_{½}}\rangle_{½} \longrightarrow \mathbf{j}_{\text{auto}\pm}^{\Psi_3}$ on each oscillating rotation axis ***directions*** $\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_k^{-1}$ is

(6.388) $\mathbf{j}_k^{\Psi_{½}} = P_{\boldsymbol{\sigma}_k}\big(\mathbf{j}_{\text{auto}\pm}^{\Psi_{½}}\big) = \big(\mathbf{j}_{\text{auto}\pm}^{\Psi_3}\cdot\boldsymbol{\sigma}_k\big)\boldsymbol{\sigma}_k^{-1} = \big(\mathbf{j}_{\text{auto}\pm}^{\Psi_3}\cdot\boldsymbol{\sigma}_k\big)\boldsymbol{\sigma}_k = \pm\hbar\frac{1}{2}\boldsymbol{\sigma}_k$,

that has the magnitude $\left|\mathbf{j}_k^{\Psi_{½}}\right| = \hbar\frac{1}{2}$. See Figure 6.22. –

How can we believe this rigid picture? It contradicts the interpretation of spin angular momentum in traditional quantum mechanics, where the origin of spin is categorised as transcendental.

The trick idea is, that we just chose the basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ and its dual quaternion basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ to represent the ***directions*** of the perpendicular angular momentum components $\mathbf{L}_k = \frac{1}{2}\boldsymbol{i}_k$ internal *autonomous* in the idea of an ***entity*** $\Psi_{½}$. This follows directly (6.300) from the geometric algebraic rules § 6.5.4 for the orthogonality idea lifted into the even real algebra $\mathbb{H} \sim \mathcal{G}_{0,2}(\mathbb{R})$.

As we have seen in this book, the idea of angular momentum for a fundamental ***entity*** $\Psi_{½}$ is synthetic endowed to the circular plane symmetry expressed in the unitary circle group $U(1)$.

For the object from the substance of each three ***directions*** $\boldsymbol{i}_k$ this idea is

(6.389) $\mathbf{L}_k^{\Psi_{½}} \leftarrow \odot_k = \{U_\theta : \theta \to e^{\boldsymbol{i}_k\theta} \in U(1) \,|\, \forall\theta \in \mathbb{R}\}$.

An external symmetry view of the total angular momentum will hide the specific ***directions*** in the spherical symmetry as displayed in Figure 6.20, where all symmetry phase factors $e^{\boldsymbol{i}\theta}$ for $\forall\theta \in [0, 2\pi[$ possesses all ***directions*** $\forall\boldsymbol{i} \in \mathfrak{Z}$ in $\mathfrak{Z}$-space described by the real even algebra $\mathcal{G}_{0,2}(\mathbb{R})$.

This total spherical symmetry we can brake by an external field possessing an angular momentum.

This can be established by an inhomogeneous magnetic field as in the Stern-Gerlach experiment.

The field gradient with a lab frame ***direction*** $\boldsymbol{\sigma}_3 = \mathbf{e}_3$ consists of free subtons that possess ***quanta*** of bivector angular momentum $\mathbf{L}_3^\pm = \pm\hbar\boldsymbol{i}_3 = \pm\hbar\boldsymbol{i}\boldsymbol{\sigma}_3 = \pm\hbar\boldsymbol{i}\mathbf{e}_3$, external to the ***entity*** $\Psi_{½}$.

This designates one symmetry ***direction***

(6.390) $\mathbf{L}_3^\pm \longrightarrow \odot_3 = \{\theta \to e^{\boldsymbol{i}_3\theta} \,|\, \forall\theta \in \mathbb{R}\} \longrightarrow \mathbf{L}_3^{\Psi_{½}} = \pm\frac{1}{2}\hbar\boldsymbol{i}_3$,

that is represents the projection of the total fluctuation angular momentum on the symmetry chosen plane ***direction*** $\boldsymbol{i}_3$, driven by the internal autonomous oscillation $U_\phi^2 = e^{\boldsymbol{i}_3\phi}$ that rotates the internal frame $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ and its dual basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ for our ***entity*** relative to our external world fixed lab frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ like (6.236)

(6.391) $\boldsymbol{\sigma}_j = U_\phi\mathbf{e}_jU_\phi^\dagger = U_\phi^2\mathbf{e}_j$, and for the ***entity*** $\Psi_{½} = U_\phi^2\Psi_{\text{lab}}$,

driven by the oscillating 1-rotor $U_\phi := e^{½\boldsymbol{i}_3\phi} \longleftarrow e^{½\boldsymbol{i}_3(\omega t + \theta)}$.

The reversed oscillating rotation of the lab, seen autonomous from the ***entity*** turns the lab frame $\mathbf{e}_j = U_\phi^\dagger\boldsymbol{\sigma}_jU_\phi$. In this, the total angular momentum rotates as

(6.392) $\mathbf{L}_\pm^{\Psi_{½}}(\phi) = U_\phi^\dagger\mathbf{L}_{\text{auto}\pm}^{\Psi_{½}}U_\phi$, and dual $\mathbf{j}_\pm^{\Psi_{½}}(\phi) = U_\phi^\dagger\,\mathbf{j}_{\text{auto}\pm}^{\Psi_{½}}U_\phi$,