only the pure scalar ground state (3.138) with ***no direction*** and no definite locality of any ℨ-space, where the tradition judge *a state of highest symmetry*.[91]

For $j > 0$ there are $2j+1$ orthogonal eigenstates. We have the quantum number m, $|m| \le j$

(6.354) $\quad j \in \mathbb{N} \Rightarrow m = \cdots, -2, -1, 0, 1, 2, \ldots,\quad$ and half-integer $\quad j \notin \mathbb{N} \Rightarrow m = \cdots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$.

The real excitation coefficients $c_\pm^* = c_\pm$ in (6.343) and (6.344) we will find by reversing

(6.355) $\quad \langle \lambda, m| = |\lambda, m\rangle^\dagger$,

(6.356) $\quad \langle \lambda, m| J_- = \langle \lambda, m+1|\, \hbar c_+(\lambda, m)$.

And combining by multiplication (6.356) on (6.343) and using (6.327) (Just as [9]p.240)

(6.357) $\quad \langle \lambda, m| J_- J_+ |\lambda, m\rangle = \hbar^2 |c_+(\lambda, m)|^2 \langle \lambda, m+1|\lambda, m+1\rangle$

$$= \langle \lambda, m| (J^2 - J_3^2 - \hbar \mathrm{j}_3) |\lambda, m\rangle = \hbar^2(\lambda - m^2 - m)\, \langle \lambda, m+1|\lambda, m+1\rangle.$$

Hence the stepping coefficients using (6.347) are positive reals

(6.358) $\quad c_\pm(\lambda, m) = \sqrt{j(j+1) - m^2 \mp m}$.

Then we find the step change eigenstates

(6.359) $\quad J_+|\lambda, m\rangle = \hbar\sqrt{(j-m)(j+m+1)}\ |\lambda, m+1\rangle,$ ([9](11.42))

(6.360) $\quad J_-|\lambda, m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\ |\lambda, m-1\rangle.$ ([9](11.43))

First when $m = \pm j$ there are no further steps.

6.5.5.3. Excitation of Angular Momentum in ℨ space

We have the quantum numbers $\boxed{j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots}$ (6.351) for quantum excitation of ℨ-space.

We skip $j = 0$ for the ground state and will below section 6.5.6 etc. look further at $j = \frac{1}{2}$.

For $j = 1$ we have three states $m = -1, 0, +1$, and $\lambda = 2$, that do ladder steps as

(6.361) $\quad J_+|2, -1\rangle \doteq \hbar\sqrt{2}|2, 0\rangle$, $\quad J_+|2, 0\rangle \doteq \hbar\sqrt{2}|2, +1\rangle$, from (6.359),

(6.361a) $\quad J_-|2, +1\rangle \doteq \hbar\sqrt{2}|2, 0\rangle$, $\quad J_-|2, 0\rangle \doteq \hbar\sqrt{2}|2, -1\rangle$, from (6.360).

There is a speciality with the state $|2, 0\rangle$ and the ***directional*** eigenstate equation (6.334)

(6.362) $\quad \mathrm{j}_3|2, 0\rangle = 0|2, 0\rangle = 0,\quad$ and its dual $\quad \mathrm{L}_3|2, 0\rangle = 0|2, 0\rangle = 0.$ (Helium-like)

Both the ***directional*** 1-vector $0\boldsymbol{\sigma}_3 = 0$ and the dual bivector $0\boldsymbol{i}_3 = 0$ represents void angular momentum $\vec{L}_k = 0$ dual $\mathrm{L}_k = 0$, therefor no specific locality center at all. Where every null-bivector $\mathrm{B}_0 = 0$ represent every plane, and null-vectors $\mathbf{0} = 0$ represents every point in ℨ-space.

Else for excitation with $m = \pm 1$, we get stats with spin one and a count of two (what two?).

(6.363) $\quad \mathrm{j}_3|2, \pm 1\rangle \doteq \pm\hbar\boldsymbol{\sigma}_3|2, \pm 1\rangle,\quad$ and its dual $\quad \mathrm{L}_3|2, \pm 1\rangle \doteq \pm\hbar\boldsymbol{i}_3|2, \pm 1\rangle.$

(6.364) $\quad J^2|2, \pm 1\rangle \doteq \hbar^2 2\, |2, \pm 1\rangle,\quad$ and its dual $\quad \mathrm{L}^2|2, \pm 1\rangle \doteq -\hbar^2 2\, |2, \pm 1\rangle.$

Next for $j = \frac{3}{2}$, we have four states $m = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$ and $\lambda = \frac{15}{4}$, that do ladder steps as

(6.365) $\quad J_+|\frac{15}{4}, -\frac{3}{2}\rangle \doteq \hbar\sqrt{3}|\frac{15}{4}, -\frac{1}{2}\rangle$, $\quad J_+|\frac{15}{4}, -\frac{1}{2}\rangle \doteq \hbar 2|\frac{15}{4}, +\frac{1}{2}\rangle$, $\quad J_+|\frac{15}{4}, +\frac{1}{2}\rangle \doteq \hbar\sqrt{3}|\frac{15}{4}, +\frac{3}{2}\rangle$ from (6.359),

(6.365a) $\quad J_-|\frac{15}{4}, +\frac{3}{2}\rangle \doteq \hbar\sqrt{3}|\frac{15}{4}, +\frac{1}{2}\rangle$, $\quad J_-|\frac{15}{4}, +\frac{1}{2}\rangle \doteq \hbar 2|\frac{15}{4}, -\frac{1}{2}\rangle$, $\quad J_-|\frac{15}{4}, -\frac{1}{2}\rangle \doteq \hbar\sqrt{3}|\frac{15}{4}, -\frac{3}{2}\rangle$ from (6.360).

Next for $j = 2$, we have five states $m = -2, -1, 0, +1, +2$, and $\lambda = 6$, that do ladder steps as

(6.366) $\quad J_+|6, -2\rangle \doteq \hbar 2|6, -1\rangle$, $\quad J_+|6, -1\rangle \doteq \hbar\sqrt{6}|6, 0\rangle$, $\quad J_+|6, 0\rangle \doteq \hbar\sqrt{6}|6, +1\rangle$, $\quad J_+|6, +1\rangle \doteq \hbar 2|6, +2\rangle$,

(6.366a) $\quad J_-|6, +2\rangle \doteq \hbar 2|6, +1\rangle$, $\quad J_-|6, +1\rangle \doteq \hbar\sqrt{6}|6, 0\rangle$, $\quad J_-|6, 0\rangle \doteq \hbar\sqrt{6}|6, -1\rangle$, $\quad J_-|6, -1\rangle \doteq \hbar 2|6, -2\rangle$,

Eight eigenstates with integer m. After this etc. $j = \frac{5}{2}, 3, \frac{7}{2}, \ldots$.

We will not go further into such excitation in this book but refer the reader to the general historical fabric of physical literature on *orbital* angular momentum excitations. Instead, we will go into the almost neglected ℨ-space structure of angular momentum of a fundamental *spin½* ***entity***.

6.5.6. The Spin½ of a *Directional Entity* of Locality in ℨ-space

6.5.6.1. The fundamental first excitation of ℨ space

When $j = \frac{1}{2}$ we have $m = \pm\frac{1}{2}$, and $\lambda = \frac{3}{4}$, refer to (6.347)

(6.367) $\quad J_+|\frac{3}{4}, -\frac{1}{2}\rangle \doteq \hbar 1|\frac{3}{4}, +\frac{1}{2}\rangle$,

(6.368) $\quad J_-|\frac{3}{4}, +\frac{1}{2}\rangle \doteq \hbar 1|\frac{3}{4}, -\frac{1}{2}\rangle$.

These stats satisfy the ***directional*** eigenvalue equation (6.334)

(6.369) $\quad \mathrm{j}_3|\frac{3}{4}, \pm\frac{1}{2}\rangle \doteq \pm\frac{1}{2}\hbar\boldsymbol{\sigma}_3|\frac{3}{4}, \pm\frac{1}{2}\rangle\quad$ and its dual $\quad \mathrm{L}_3|\frac{3}{4}, \pm\frac{1}{2}\rangle \doteq \pm\frac{1}{2}\hbar\boldsymbol{i}_3|\frac{3}{4}, \pm\frac{1}{2}\rangle$.

By left multiplying with the reversed state $\langle\frac{3}{4}, \pm\frac{1}{2}|$ we find the average[351] in this ***direction***

(6.370) $\quad \langle\frac{3}{4}, \pm\frac{1}{2}|\, \mathrm{L}_3\, |\frac{3}{4}, \pm\frac{1}{2}\rangle = \langle\frac{3}{4}, \pm\frac{1}{2}|\, \pm\frac{1}{2}\hbar\boldsymbol{i}_3\, |\frac{3}{4}, \pm\frac{1}{2}\rangle \Longrightarrow \langle \mathrm{L}_3\rangle_{½} = \pm\frac{1}{2}\hbar\boldsymbol{i}_3$, and dual $\langle \mathrm{j}_3\rangle_{½} = \pm\frac{1}{2}\hbar\boldsymbol{\sigma}_3$.

This has the real scalar eigenvalues $\langle\lambda_3\rangle = \pm\frac{1}{2}$ for $\hbar = 1$, that is the same as (6.300) $\lambda_3 = \pm\frac{1}{2}$.

These stats also satisfy the ***no directional*** scalar eigenvalue equation (6.335)

(6.371) $\quad J^2|\frac{3}{4}, \pm\frac{1}{2}\rangle \doteq \hbar^2\frac{3}{4}\, |\frac{3}{4}, \pm\frac{1}{2}\rangle$.

The expectation average value of this $\langle J^2\rangle_{½} = \hbar^2\frac{3}{4}$.

Here we remember from (6.322) and (6.320)

(6.372) $\quad J^2 = \mathrm{j}_1^2 + \mathrm{j}_2^2 + \mathrm{j}_3^2 = \hbar^2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \ge 0.$

Due to the symmetry, we can write $\langle\lambda_k\rangle_{½} = \pm\frac{1}{2}$ and squared $\langle\lambda_k^2\rangle_{½} = \frac{1}{4}$.

Combining Pythagorean square, we get the squared radial magnitude of angular momentum

(6.373) $\quad |\langle J^2\rangle_{½}| = \langle\lambda_r^2\rangle_{½} = \lambda = \frac{3}{4}$

We compare this with (6.299) $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \frac{3}{4}$ for an ***entity*** $\Psi_{½}$ that fulfils the even closed quaternion algebra $\mathbb{H} \sim \mathcal{G}_{0,2}(\mathbb{R})$ and find that the eigenstate $\psi_\pm^{½} = |\frac{3}{4}, \pm\frac{1}{2}\rangle$ is a quaternion 2-spinor supported from the basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ for the even closed quaternion group (6.130).

This shows: The geometric algebra $\mathbb{H} \sim \mathcal{G}_{0,2}(\mathbb{R})$ is the foundation for the first excitation state $j = \frac{1}{2}$ (6.351) of angular momenta around a locality center ***entity*** $\Psi_{½}$.

Such a state is called a *spin½ state* of an ***entity*** in a physical ℨ-space. Since Leibniz, it has been discussed that different extended solids cannot possess the same locality in space. This was re-formulated by Pauli to the exclusion principle for the electron stats. Here we will just presume that distinguishable ***entities*** $\Psi_{½}$ do not possess the same locality center in space.

This fundamental ***quality*** of ***entities*** $\Psi_{½}$ is in the tradition of ***quantum*** mechanics called *Fermi particles*. This is the ***quality*** we call an exclusive central locality with an internal ontological existence. Complementary expressed as the external S^2 symmetry sphere in ℨ-space. Some call these centres for *point-particles* as an a priori transcendental concept without any reality existence (no extension) and by that, no ***directions*** in space at all, as for geometrical points of pure ***primary quality of zero grade***, that therefore not a physical ***entity***, only the eternal ground state. Intuitively spoken a pure geometric point cannot possess any angular momentum.

6.5.6.2. Symmetry Braking of the Half Spin Ψ½ Entity,

Interaction by external subtons $\hat{\phi}_{3\pm} = |\pm 1\rangle_3 = e^{\pm \boldsymbol{i}_3\phi}$ (6.229) in the lab ***direction*** $\boldsymbol{\sigma}_3 = \mathbf{e}_2$ with quantised angular momentum $\mathrm{L}_3^\pm = \pm\hbar\boldsymbol{i}_3$, where $\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3$, can be created e.g., by an external magnetic field gradient. This may alter the spherical symmetry of $\Psi_{½}$ ***entities***

(6.374) $\quad \langle\lambda_r^2\rangle_{½} = \langle\, \lambda_1^2 + \lambda_2^2\, \rangle_{½} + \langle\lambda_3^2\rangle_{½} = \frac{3}{4} = \frac{1}{2} + \frac{1}{4} \iff \langle\, \lambda_1^2 + \lambda_2^2\, \rangle_{½} = \frac{1}{2}$

In this state $|\frac{3}{4}, \pm\frac{1}{2}\rangle$ of the ***directional entity*** $\Psi_{½} = \Psi_3$, we have an angular momentum

[351] Here in this context, we preferer auto-normalization $\langle\frac{3}{4}, \pm\frac{1}{2}|\frac{3}{4}, \pm\frac{1}{2}\rangle = 1$, just as $\hbar = 1$, when it comes to timing measure $|\omega| = 1$, $c = 1$.

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