By making use of $J^2 \doteqdot \mathbf{j}^2$ we do not get rid of the mix of a scalar and a 1-vector in the operator

(6.330) $$J_\pm J_\mp = J^2 - J_3^2 \pm \hbar \mathbf{j}_3 = J_\perp^2 \pm \hbar \mathbf{j}_3. \qquad \sim \langle A\rangle_0 + \langle A\rangle_1$$

The total scalar operator $\mathbf{L}^2, \mathbf{j}^2$ and by that J^2 commutes with the other operators

(6.331) $$[J^2, \mathbf{j}] = J^2\mathbf{j} - \mathbf{j}\,J^2 = 0, \qquad [J^2, \mathbf{j}_k] = 0, \qquad [J^2, J_\pm] = 0.$$

(6.332) $$[\mathbf{j}^2, \mathbf{j}] = \mathbf{j}^2\mathbf{j} - \mathbf{j}\,\mathbf{j}^2 = 0, \qquad [\mathbf{j}^2, \mathbf{j}_k] = 0, \qquad [\mathbf{j}^2, J_\pm] = 0.$$

And their dual operators also commute with these scalar operators

(6.333) $$[\mathbf{L}^2, \mathbf{L}] = \mathbf{L}^2\mathbf{L} - \mathbf{L}\mathbf{L}^2 = 0, \qquad [\mathbf{L}^2, \mathbf{L}_k] = 0, \qquad [\mathbf{L}^2, J_\pm] = 0.$$

6.5.5. The Quantum Stats of the Locally Combined Angular Momentum

We now set a symbol for the total eigenstates $|\lambda, m\rangle$ for the local ***entity*** Ψ_3 described by the two-angular-parameter S^2 symmetry as arguments for quaternion 2-spinors in the even $\mathcal{G}_{0,2}(\mathbb{R})$ geometric algebra for a $\mathfrak{Z}$-space, as described in section 6.4.9.
We interpret m and λ as real eigenvalue ***quantisation*** numbers for angular momenta of the $\mathcal{G}_{0,2}$ spinors expressed as $|\lambda,m\rangle \leftrightarrow \Psi_3$ for the local space ***entity***.
The bindings for these stats we establish by ***directional*** bivector eigenvalue equations.[349]
First for ***direction***, as we have seen earlier, we simulate the fundamental circle oscillators

(6.334) $$\mathbf{j}_3|\lambda,m\rangle \doteq \hbar m \boldsymbol{\sigma}_3 |\lambda,m\rangle, \qquad \text{and its dual} \qquad \mathbf{L}_3|\lambda,m\rangle \doteq \hbar m \boldsymbol{i}_3 |\lambda,m\rangle.$$

quality of first grade (***pqg***-1), - ***quality of second grade*** (***pqg***-2) $\to \mathcal{G}_{0,2}$.

And further the eigenvalue equation for the squared scalar operators J^2, alternative $\mathbf{L}^2$ both for ***no direction quality of zero grade***, ***pqg***-0 $\to \mathcal{G}_{0,2}$ in the even geometric algebra

(6.335) $$J^2|\lambda,m\rangle \doteq \hbar^2\lambda\,|\lambda,m\rangle \qquad \text{and its dual} \qquad -\mathbf{L}^2|\lambda,m\rangle \doteq \hbar^2\lambda\,|\lambda,m\rangle.$$

These two different operators $\mathbf{j}_3$ and J^2 commute, hence they have the same eigenstates $|\lambda,m\rangle$, and that is the same for their duals $\mathbf{L}_3$ and $\mathbf{L}^2$.
We first ignore the total operator $\mathbf{j}$ or $\mathbf{L}$ and their specific components $\mathbf{j}_1, \mathbf{j}_2$ and $\mathbf{j}_\perp$ or $\mathbf{L}_1, \mathbf{L}_2$ and $\mathbf{L}_\perp$ as implicit because they are transcendental to our intuition and knowledge.
In general, we demand that an *expectation value* of an operator B is positive

(6.336) $$|B^2| = |B|^2 = B^\dagger B \geq 0.$$

This is the case for multivectors if we use $B^\dagger = \tilde{B}$ for the reversed sequential operation order.
When we operate on ***quantum*** states $|\lambda,m\rangle$, this is expressed as

(6.337) $$\langle\lambda,m|B^\dagger B|\lambda,m\rangle = (B|\lambda,m\rangle)^\dagger B|\lambda,m\rangle = \big|B|\lambda,m\rangle\big|^2 \geq 0.$$

Hereby we in agreement with (6.323)-(6.325) for the squared perpendicular conclude

(6.338) $$\langle\lambda,m|J^2 - J_3^2|\lambda,m\rangle = \langle\lambda,m|\mathbf{L}_3^2 - \mathbf{L}^2|\lambda,m\rangle = \tfrac{1}{2}\left\langle\lambda,m\middle|J_+J_+^\dagger\middle|\lambda,m\right\rangle + \tfrac{1}{2}\left\langle\lambda,m\middle|J_+^\dagger J_+\middle|\lambda,m\right\rangle \geq 0.$$

From the two eigenstates equations (6.335), and (6.334) squared, we find

(6.339) $$\langle\lambda,m|J^2 - J_3^2|\lambda,m\rangle = \langle\lambda,m|\mathbf{L}_3^2 - \mathbf{L}^2|\lambda,m\rangle \sim \hbar^2\lambda - \hbar^2 m^2 \geq 0 \quad\Longrightarrow\quad \lambda \geq m^2 \geq 0.$$

Due to, that the squared operators as scalars have no ***direction***, they commute with all the multivector operators and they all have the same eigenstates $|\lambda,m\rangle$, we try to do ladder operation[350] by now using the stepping multivectors operators $J_\pm$.

[349] To understand the ***directional*** eigenvalues consult § 6.4.9.1 formula (6.229)-(6.232).
[350] Just as in section I. 3.1.3 further expanded for the circle oscillator in section I. 3.3. as $a_{\odot\pm}^\dagger$

6.5.5.2. The Quantum Ladder Step Operations in $\mathfrak{Z}$ space

We presume the changed state $J_\pm|\lambda,m\rangle = J_\mp^\dagger|\lambda,m\rangle$ is the same as annihilation\creation of synchronised oscillators fulfilling (6.334), as we use these eigenvalue equations on this state and apply (6.311)-(6.314) to do the steps (refer to [9]p.239)

(6.340) $$\mathbf{L}_3 J_+|\lambda,m\rangle \doteq \hbar(m+1)J_+|\lambda,m\rangle \qquad \text{or its dual} \qquad \mathbf{j}_3 J_+|\lambda,m\rangle \doteq \hbar(m+1)J_+|\lambda,m\rangle.$$

(6.341) $$\mathbf{L}_3 J_-|\lambda,m\rangle \doteq \hbar(m-1)J_-|\lambda,m\rangle. \qquad \text{or its dual} \qquad \mathbf{j}_3 J_-|\lambda,m\rangle \doteq \hbar(m-1)J_-|\lambda,m\rangle$$

and that also fulfil (6.335) due to the commutation relations (6.331)-(6.333)

(6.342) $$J^2 J_\pm|\lambda,m\rangle \doteq \hbar^2\lambda\,J_\pm|\lambda,m\rangle.$$

These exited stats (annihilation\creation) we try to estimate as

(6.343) $$J_+|\lambda,m\rangle \doteq \hbar c_+(\lambda,m)\,|\lambda,m+1\rangle,$$

(6.344) $$J_-|\lambda,m\rangle \doteq \hbar c_-(\lambda,m)\,|\lambda,m-1\rangle.$$

These scalar factors $c_\pm(\lambda,m)$ can be complex numbers, we choose to see them as reals, and put the uncertainty phase factor into the plane angular rotation symmetry $\odot_{i_3} = \{\theta \to e^{i_3\theta} | \forall\theta \in \mathbb{R}\}$.
From (6.339) $\lambda \geq m^2$ we see that m have its greatest value $j = \mathrm{Max}(m) \geq 0$ for any given $\lambda \geq 0$.
In all cases for a fundamental ***entity*** Ψ_3 (an atomic element), we presume a priori that λ is finite (reasonably small), which thus assumed $j^2 \lesssim \lambda$
In the max case $m = j$ an extra step with the same orientation is beyond the limit, therefore, is demanded as void, thus we chose

(6.345) $$J_+|\lambda, j\rangle = 0$$

From this zero, the opposite orientation ladder step operator J_- is recreating the state $|\lambda, j\rangle$ where we from (6.330) get the eigenvalue equation ($\doteq$) that results in a total eigenvalue combination

(6.346) $$J_-J_+|\lambda,j\rangle = (J^2 - J_3^2 - \hbar\mathbf{j}_3)\,|\lambda,j\rangle \doteq (\lambda - j^2 - j)\,|\lambda,j\rangle = J_- 0 = 0.$$

This demands that for the possible j we can expect

(6.347) $$\lambda = j(j+1).$$

For the lowest case $j' = \mathrm{Min}(m)$ an extra step with the same orientation is void $J_-|\lambda, j'\rangle = 0$

(6.348) $$J_+J_-|\lambda,j'\rangle = (J^2 - J_3^2 + \hbar\mathbf{j}_3)\,|\lambda,j\rangle \doteq \left(\lambda - j'^2 + j'\right)|\lambda,j'\rangle = J_+ 0 = 0.$$

This demands that for the possible j' we can expect

(6.349) $$\lambda = j'(j'-1).$$

Because we demand $j' \leq j$ (6.347) combined with (6.349) we obtain $j' = -j$.
We here have two extreme states: top $|\lambda, j\rangle$ and bottom $|\lambda, -j\rangle$, that fulfil eigenvalue equations $J_-J_+|\lambda, j\rangle \doteq 0|\lambda, j\rangle$, and $J_+J_-|\lambda,-j\rangle \doteq 0|\lambda,-j\rangle$.
The natural number of states $|\lambda,m\rangle$ with $[-j \leq m \leq j]$ from bottom to top is $2j+1 \geq 1$.
All the $n = 2j$ steps down and then all the $n = 2j$ steps up

(6.350) $$(J_-)^n|\lambda,j\rangle \propto |\lambda,j-n\rangle = |\lambda,-j\rangle \qquad \text{and} \qquad (J_+)^n|\lambda,-j\rangle \propto |\lambda,j\rangle.$$

The state quantum number m step by integer values trough $2j+1 \in \mathbb{N}$ cases, as (6.343),(6.344).
Hence, j can take nonnegative half-integer values

(6.351) $$\boxed{j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots}.$$

This will govern the eigenvalues to the ***direction*** operator $\mathbf{L}_3$ of the eigenvalue equation (6.334)

(6.352) $$\mathbf{L}_3|\lambda,m\rangle \doteq \hbar m \boldsymbol{\sigma}_3|\lambda,m\rangle.$$

The real scalar ***quantum*** eigenvalue can appear as

(6.353) $$\hbar m = \hbar j, \ \hbar(j-1), \ \hbar a(j-2), \ \ldots, \ -\hbar(j-2), \ -\hbar(j-1), \ -\hbar j.$$

where we note that $\hbar m \neq 0$, for $j \notin \mathbb{N}$, and special $j=0 \Rightarrow m=0$ with no existence of an ***entity*** Ψ_3,