(6.288) $[\mathbf{L}_1,\mathbf{L}_2] = (\hbar\lambda_1\boldsymbol{i}\sigma_1)(\hbar\lambda_2\boldsymbol{i}\sigma_2) - (\hbar\lambda_2\boldsymbol{i}\sigma_2)(\hbar\lambda_1\boldsymbol{i}\sigma_1) = \hbar^2 2\lambda_1\lambda_2\sigma_2\sigma_1 = \hbar^2 2\lambda_1\lambda_2\boldsymbol{i}_3 \sim \hbar^2\lambda_3\boldsymbol{i}_3 = \hbar\mathbf{L}_3.$

This bivector relation $[\mathbf{L}_1,\mathbf{L}_2] = \hbar\mathbf{L}_3$ exist purely in the even closed $\mathbb{H}\sim\mathcal{G}_{0,2}(\mathbb{R})$ geometric algebra and expresses the interconnected relations of the different ***pqg*-2 *directions qualities*** of $\mathfrak{Z}$-space. In a short form of angular momentum bivector multiplication, we just write

(6.289) $\boxed{\mathbf{L}_1\mathbf{L}_2 = \hbar\tfrac{1}{2}\mathbf{L}_3 \quad = \mathrm{j}_2\mathrm{j}_1 = \hbar\boldsymbol{i}\tfrac{1}{2}\mathrm{j}_3}$ and reversed $\boxed{\mathbf{L}_2\mathbf{L}_1 = -\hbar\tfrac{1}{2}\mathbf{L}_3 \quad = \mathrm{j}_1\mathrm{j}_2 = -\hbar\boldsymbol{i}\tfrac{1}{2}\mathrm{j}_3}$.

These expressions (6.285)-(6.289) relates multivectors of a geometric algebra $\mathcal{G}_3$.

By correspondence[337] $i \leftrightarrow \boldsymbol{i} \sim \sqrt{-1}$ to traditional quantum mechanics, e.g., [9](11.5)-(Merzbacher), the orthogonal angular momentum operator commutator relations are written as (3.77):

(6.290) $[\hat{L}_1,\hat{L}_2] = i\hbar\hat{L}_3, \qquad [\hat{L}_2,\hat{L}_3] = i\hbar\hat{L}_1, \qquad [\hat{L}_3,\hat{L}_1] = i\hbar\hat{L}_2.$

6.5.2. The Commutator Relations in Geometric Algebra for Angular Momenta

For the three basic orthogonal operator ***directions*** of angular momenta, we have interconnectivity

Commutators			Geometric Products		
	pqg-2, *pqg*-2 → *pqg*-2	*pqg*-1, *pqg*-1 → *pqg*-2		*pqg*-2, *pqg*-2 → *pqg*-2	*pqg*-1, *pqg*-1 → *pqg*-2
(6.291)	$[\mathbf{L}_1,\mathbf{L}_2] = \hbar\mathbf{L}_3$	$= [\mathbf{j}_2,\mathbf{j}_1] = \hbar\boldsymbol{i}\,\mathbf{j}_3$	(6.291a)	$\mathbf{L}_1\mathbf{L}_2 = \hbar\frac{1}{2}\mathbf{L}_3$	$= \mathbf{j}_2\mathbf{j}_1 = \hbar\boldsymbol{i}\frac{1}{2}\mathbf{j}_3$
(6.292)	$[\mathbf{L}_2,\mathbf{L}_3] = \hbar\mathbf{L}_1$	$= [\mathbf{j}_3,\mathbf{j}_2] = \hbar\boldsymbol{i}\,\mathbf{j}_1$	(6.292a)	$\mathbf{L}_2\mathbf{L}_3 = \hbar\frac{1}{2}\mathbf{L}_1$	$= \mathbf{j}_3\mathbf{j}_2 = \hbar\boldsymbol{i}\frac{1}{2}\mathbf{j}_1$
(6.293)	$[\mathbf{L}_3,\mathbf{L}_1] = \hbar\mathbf{L}_2$	$= [\mathbf{j}_1,\mathbf{j}_3] = \hbar\boldsymbol{i}\,\mathbf{j}_2$	(6.293a)	$\mathbf{L}_3\mathbf{L}_1 = \hbar\frac{1}{2}\mathbf{L}_2$	$= \mathbf{j}_1\mathbf{j}_3 = \hbar\boldsymbol{i}\frac{1}{2}\mathbf{j}_2$

This is an a priori concept idea for ***quantum*** operations of angular momentum that possesses the ***primary qualities of directions***. Two orthogonal angular momenta ***directions*** do not commute, but the interesting thing is, that all these *orthogonal operator components anticommute*

(6.294) $\mathrm{j}_k\mathrm{j}_j + \mathrm{j}_j\mathrm{j}_k = 0$ and dual $\mathbf{L}_j\mathbf{L}_k + \mathbf{L}_k\mathbf{L}_j = 0$ where $k \neq j$, for $j,k = 1,2,3$..

We simply see this by their definition, e.g., by the anti-version of (6.285)

(6.295) $\mathrm{j}_2\mathrm{j}_1 + \mathrm{j}_1\mathrm{j}_2 = \hbar^2\lambda_1\lambda_2\sigma_2\sigma_1 + \hbar^2\lambda_1\lambda_2\sigma_1\sigma_2 = \hbar^2\lambda_1\lambda_2(\sigma_2\sigma_1 + \sigma_1\sigma_2) = 0.$

The advantage of orthogonality for the algebraic structure of the geometric perpendicular planes for the angular momenta is, that it makes the scalar ***quality pqg*-0** immaterial.[338]
The idea of angular momenta expressed as *bivectors* interact by a product resulting in a new *bivector* that stays inside the even closed algebra $\mathbb{H}\sim\mathcal{G}_{0,2}(\mathbb{R})$, and their interconnectivity is expressed by the commutator product purely as the ***primary quality of second grade*** (***pqg***-2).
If we use the traditional ***first grade*** (***pqg***-1) picture of the angular momentum 1-vector, we must involve the ***third grade*** (***pqg***-3) chiral volume pseudoscalar $\boldsymbol{i}$ in the interconnectivity commutator relation to staying inside the odd algebra. The product of two angular momentum ***pqg***-1-vectors are direct lifted into the even algebra where their further product results will stay closed as ***pqg***-2 bivectors when we stick to orthogonality.
As an a priori idea, angular momentum is a ***pqg*-2 *direction quality*** internal in a plane concept.[339]

[337] Is worth for the reader to note the different sequential order notation for operators. In this book, we prefer operation interpretation from right to left of the sequential operations, special when concerning multivector products.
We remember the sequence definitions for the plane pseudoscalar $\boldsymbol{i}_3 \equiv \sigma_2\sigma_1$ and the chiral pseudoscalar $\boldsymbol{i} \equiv \sigma_3\sigma_2\sigma_1$.
In traditional QM we often see $-\hbar i$ written as $\hbar/i$, in its principle, it is 'only' a reversed order.
[338] The concept of angular momentum as a bivector, hides implicit the substance of a physical circle oscillator, as the generator is expressed as an oscillating rotor e.g., $e^{\pm\boldsymbol{i}_3 \frac{1}{2}\phi}$, where the scalar part of the oscillation as term cos $\frac{1}{2}\phi$, $(\phi = \omega t)$ is hidden.
[339] Kepler's second law.

As soon as it comes to the $\mathfrak{Z}$-space concept, the even closed algebra $\mathbb{H}\sim\mathcal{G}_{0,2}(\mathbb{R})$ does the work of interconnectivity entanglement internal for an ***entity*** Ψ_3.
But the dual odd algebra ***pqg***-1,3 with the 1-vector representation of the angular momentum components may help us by the chirality volume pseudoscalar $\boldsymbol{i}$ to analyse the ***direction*** interaction with the external surroundings.

6.5.2.2. Orthogonal Chirality of Angular Momenta in the Even $\mathcal{G}_{0,2}$ Geometric Algebra

As expressed in the quaternion picture (6.127)-(6.130) the bivector basis $\{\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\}$ has the two opposite orientations of the triple product $\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -1$ and $\boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1 = +1$ as the two unit scalar orientations inside the closed even 0+2-***grade*** quaternion group with standard basis $\{1,\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\}$.
The same chiral triple orthogonality rule we deducted for the closed interconnectivity of the three angular momentum operator components, in that we multiply $\mathbf{L}_1\mathbf{L}_2 = \hbar\frac{1}{2}\mathbf{L}_3$ (6.289) by $\mathbf{L}_3$ getting

(6.296) $\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3 = \hbar\tfrac{1}{2}\mathbf{L}_3\mathbf{L}_3 = \hbar\tfrac{1}{2}\mathbf{L}_k^2 = \hbar^3\lambda_1\lambda_2\lambda_3\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -\hbar^3\lambda_1\lambda_2\lambda_3 = -\hbar^3\tfrac{1}{2}\lambda_k^2,$ for $k = 1,2,3$.

etc. with equivalent for (6.291a)-(6.293a).
For the reversed order in the chiral angular volume, we have

(6.297) $\mathbf{L}_3\mathbf{L}_2\mathbf{L}_1 = -\hbar\tfrac{1}{2}\mathbf{L}_k^2 = \hbar^3\tfrac{1}{2}\lambda_k^2$

This symmetric interconnectivity between the bivector ***directions*** leads to $\lambda_1^2 = \lambda_2^2 = \lambda_3^2$, and using the fit demand $\boxed{\lambda_3 = 2\lambda_1\lambda_2 \;\Rightarrow\; \lambda_k = \pm 2\lambda_k^2 \;\Rightarrow\; \lambda_k = \pm\tfrac{1}{2}}$. This implis

(6.298) $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \tfrac{1}{4},$

in this symmetric case and we further remark

(6.299) $\lambda_k\lambda_k = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \tfrac{3}{4}.$

This (6.298) explicit the fact

(6.300) $\lambda_1 = \pm\tfrac{1}{2}, \quad \lambda_2 = \pm\tfrac{1}{2}, \quad \lambda_3 = \pm\tfrac{1}{2}.$

We presume the existence of an ***entity*** $\Psi_{½}$ for the positive case, which defines the operators

(6.301) $\mathbf{L}_1 = \hbar\tfrac{1}{2}\boldsymbol{i}_1, \quad \mathbf{L}_2 = \hbar\tfrac{1}{2}\boldsymbol{i}_2,$ and $\mathbf{L}_3 = \hbar\tfrac{1}{2}\boldsymbol{i}_3,$ (autonomous for an ***entity*** $\Psi_{½}$, $\hbar=1$).

In all, the interconnected orthogonal triple bivector product interaction results in a scalar

(6.302) $\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3 = -\hbar^3\tfrac{1}{8},$ and reversed $\mathbf{L}_3\mathbf{L}_2\mathbf{L}_1 = +\hbar^3\tfrac{1}{8}.$

By this simple expression, we see that the reversed order in the ***directions*** of the angular momentum operations changes the chiral volume spin orientation just as the change of orientation for a single component e.g., $\mathbf{L}_3 = \pm\hbar\frac{1}{2}\boldsymbol{i}_3$ by the scalar amount $\pm\hbar 1$.
The triple product (6.302) of orthogonal angular momentum bivector components describes the fundamental volume symmetry of an indivisible (atomic-element) ***entity*** $\Psi_{½}$ in $\mathfrak{Z}$-space, just as the model (6.127)-(6.128) of ***grade***-2→0 in the even geometric algebra $\mathcal{G}_{0,2}(\mathbb{R})$ for quaternions $\mathbb{H}$ in section 6.4.3.