6.5.1.3. The Angular Momentum Operator

In 3-space we then write[332] the traditional angular momentum operator

(6.266) $\hat{L} = \vec{q} \times \frac{\hbar}{i}\nabla = -\vec{q} \times i\hbar\nabla = -i\hbar(\vec{q} \times \nabla).$

Equivalent to this we define the bivector multiplication operator for angular momentum[333]

(6.267) $\mathbf{L} = \hbar\mathbf{q}\wedge\nabla = \hbar\left(q_2\frac{\partial}{\partial q_3} - q_3\frac{\partial}{\partial q_2}\right)\boldsymbol{i}_1 + \hbar\left(q_3\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_3}\right)\boldsymbol{i}_2 + \hbar\left(q_1\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial q_1}\right)\boldsymbol{i}_3.$

Dual to this bivector operator we also have a 1-vector operator for angular momentum

(6.268) $\mathbf{j} = -\boldsymbol{i}\mathbf{L} = -\boldsymbol{i}\,\hbar\mathbf{q}\wedge\nabla \quad\leftrightarrow\quad \hat{L}$

Using the correspondence principle for the traditional $i\sim\sqrt{-1}$ in QM, and[334] the unit chirality volume pseudoscalar $\boldsymbol{i}\sim\sqrt{-1}$ we associate the j^{th} ***directions*** $\boldsymbol{\sigma}_j$ with duality $\boldsymbol{i}_j$ for momentum

(6.269) $\vec{p} = \mathbf{p} = p_1\boldsymbol{\sigma}_1+p_2\boldsymbol{\sigma}_2+p_3\boldsymbol{\sigma}_3,$

as operator components using the plane pseudo scalar ***quality*** $\boldsymbol{i}_j\sim\sqrt{-1}$

(6.270) $\hat{p}_1 \leftrightarrow \boldsymbol{i}\boldsymbol{\sigma}_1\hbar\frac{\partial}{\partial q_1} = \boldsymbol{i}_1\hbar\frac{\partial}{\partial q_1},\qquad \hat{p}_2 \leftrightarrow \boldsymbol{i}\boldsymbol{\sigma}_2\hbar\frac{\partial}{\partial q_2} = \boldsymbol{i}_2\hbar\frac{\partial}{\partial q_2},\qquad \hat{p}_3 \leftrightarrow \boldsymbol{i}\boldsymbol{\sigma}_3\hbar\frac{\partial}{\partial q_3} = \boldsymbol{i}_3\hbar\frac{\partial}{\partial q_3}.$

The reader may recall the correspondence with ladder operators (3.11)-(3.13)

(6.271) $a_j \coloneqq \frac{1}{\sqrt{2}}\left(q_j + \frac{\partial}{\partial q_j}\right)$ and $a_j{}^\dagger \coloneqq \frac{1}{\sqrt{2}}\left(q_j - \frac{\partial}{\partial q_j}\right).$

Then for the 3 orthogonal ***directions*** and by (3.87)

(6.272) $\hat{q}_j = \frac{1}{\sqrt{2}}(a_j + a_j{}^\dagger) \sim q_j$ and $\hat{p}_j = \boldsymbol{i}_j\frac{1}{\sqrt{2}}(a_j{}^\dagger - a_j) \sim \frac{1}{\hbar}p_j$ or $\frac{\partial}{\partial q_j} = \frac{1}{\sqrt{2}}(a_j - a_j{}^\dagger).$

And further for the circular oscillations the lather operators (3.92)-(3.95)

(6.273) $a_{k\pm} \coloneqq \frac{1}{\sqrt{2}}(a_i \mp \boldsymbol{i}a_j)$ and $a_{k\pm}^\dagger \coloneqq \frac{1}{\sqrt{2}}(a_i^\dagger \pm \boldsymbol{i}a_j^\dagger),$ cyclic: $i,j,k = 1,2,3.$

etc. … section I. 3.2 - 3.3.

In all, we for the angular momentum have the correspondence ($\hbar$=1)

(6.274) $\hat{L} = \vec{q}\times\frac{\hbar}{i}\nabla \quad\leftrightarrow\quad \mathbf{L} = \hbar\mathbf{q}\wedge\nabla = \hbar(\lambda_1\boldsymbol{i}_1+\lambda_2\boldsymbol{i}_2+\lambda_3\boldsymbol{i}_3)$

The substance of the bivector operator $\mathbf{q}\wedge\nabla$ is, that it just expresses the foundation of Kepler's second law, that in the tradition, says: A line segment $\mathbf{q}$ joining a planet and the Sun sweeps out equal angular areas $\mathbf{q}\wedge\nabla$ per chronometric measure.

6.5.1.4. Excitation of Angular Momentum in three directions of 3 Space

In § 6.4.9.1 (6.226) we recalled the ***quantum*** excitation of one *free* circle oscillation $\hat{L}_3|1,\pm1\rangle \doteq \pm1\hbar|1,\pm1\rangle$, with an eigenvalue as a 1-vector $\vec{L}_3^{\pm}$ for the angular momentum of intern auto defined magnitude $|\vec{L}_3^{\pm}| = \hbar = 1$. This given 1-vector $\vec{L}_3^{+}$ ***direction*** we use to define the autonomic frame ***direction*** $\boldsymbol{\sigma}_3 \coloneqq \vec{L}_3^{+}$, which implies the dual frame plane ***direction*** (6.235)

(6.275) $\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3 = \mathrm{L}_3^{+} \coloneqq \boldsymbol{i}\vec{L}_3^{+},$ as a transversal bivector for the free angular momentum L_3^{+}.

The idea of an orthonormal frame $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ for an 3-space ***entity*** gives the *idea* of three *free* angular momentum excitation of isometric autonomic ***quanta*** $|\vec{L}_1^{\pm}| = |\vec{L}_2^{\pm}| = |\vec{L}_3^{\pm}| = \hbar = 1,$

(6.276) $\boldsymbol{\sigma}_1 = \vec{L}_1^{+},\qquad \boldsymbol{\sigma}_2 = \vec{L}_2^{+},$ and the first chosen ***direction*** $\boldsymbol{\sigma}_3 \coloneqq \vec{L}_3^{+}.$

With the use of the unit chiral volume $\boldsymbol{i}$, we get the dual $\mathbb{H}\sim\mathcal{G}_{0,2}(\mathbb{R})$ bivector basis $\{1,\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\}$, which encourages us to define three free angular momentum ***directions*** each of ***one quantum***

[332] This idea is taken from Merzbacher [9](11.3),p.233ff. and I. (3.69),(3.70) , but the new is the transversal bivector formulation.

[333] We still use blue colour to indicate an operator. When it comes to multivector operators we only use the hat^ symbol for units.

[334] The two objects are not the same although they both squares to the same negative scalar unit −1. They by all means belong to two different narratives for an ontology of physical ***entities***.

(6.277) $\mathrm{L}_1^{+} = \boldsymbol{i}_1,\qquad \mathrm{L}_2^{+} = \boldsymbol{i}_2$ and the first chosen $\mathrm{L}_3^{+} = \boldsymbol{i}_3,$

for the fictive states $|{+1}\rangle_1$ and $|{+1}\rangle_2$ joining the considered state $\hat{\phi}_+ = |{+1}\rangle_3 \sim e^{\boldsymbol{i}_3\phi}$, (6.241).

The idea is so simple, that the three angular momentum bivectors are perpendicular

(6.278) $L_3^{+} \perp L_2^{+} \perp L_1^{+} \perp L_3^{+},$

when we try to describe the local ***directional*** autonomy of an ***entity*** in 3-space.

The interconnectivity expressed by (6.123) and (6.126) demands that the two other states $|{+1}\rangle_1$ and $|{+1}\rangle_2$ are mixed. In the traditional ***quantum mechanics*** (QM), this is expressed as 1-vector operator commutator relation $[\hat{L}_1,\hat{L}_2] = i\hbar\hat{L}_3$ in QM, (3.77) and e.g. Merzbacher [9](11.5).

Here in the even closed geometric algebra $\mathcal{G}_{0,2}(\mathbb{R})$ with basis (6.126), $\{1,\ \boldsymbol{i}_1,\ \boldsymbol{i}_2,\ \boldsymbol{i}_3 \coloneqq \boldsymbol{i}_1\boldsymbol{i}_2\}$ it is expressed as angular momentum bivector operators (without hats ^)

(6.279) $\mathrm{L}_1 = \hbar\lambda_1\boldsymbol{i}_1,\qquad \mathrm{L}_2 = \hbar\lambda_2\boldsymbol{i}_2,$ and $\mathrm{L}_3 = \hbar\lambda_3\boldsymbol{i}_3,$ together with a scalar $\hbar\lambda_0 = \hbar\lambda_0 1.$

We intuit comprehend the component 'coordinates' $\forall\lambda_\mu\in\mathbb{R}$ as real scalar operators[335] that act on the three individual plane unit pseudoscalars $\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3$, which are *interconnected* (entangled) as a generating bivector basis for the ***entity*** Ψ_3 in 3-space as an *even closed* quaternion algebra $\mathcal{G}_{0,2}$ with the quaternion basis $\{1,\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\}$.

– Besides, dual to this, we too have its *odd open* algebra of ***pqg***-1-vector perpendicular ***directions*** of the angular momentum ***pqg***-1-vector *operators*

(6.280) $\mathrm{j}_1 = \hbar\lambda_1\boldsymbol{\sigma}_1,\qquad \mathrm{j}_2 = \hbar\lambda_2\boldsymbol{\sigma}_2,$ and $\mathrm{j}_3 = \hbar\lambda_3\boldsymbol{\sigma}_3.$

The reader should notice that the determent ***directions*** are linked by the *odd* basis $\{\boldsymbol{\sigma}_k\}$ or the *even* basis $\{1,\boldsymbol{i}_k\}$ and the angular momenta variable magnitudes are expressed through the three component scalar operators $\lambda_k\in\mathbb{R},\ k=1,2,3$ for the orthogonal ***directions***.

The even algebra scalar λ_0 is just without ***direction***.

6.5.1.5. Commutator Products of Angular Momentum Bivectors and their Dual 1-vectors

In Geometric Algebra, we define the generalised commutator product of two multivectors

(6.281) $A\times B = \frac{1}{2}(AB - BA),$ introduced (6.55) and § 6.2.5.6

In quantum mechanics, we define the commutator products of two operators A and B as

(6.282) $[A,B] = AB - BA,$ introduced (2.51)

Translating this to geometric algebra we write

(6.283) $[A,B] = 2[A\times B] = AB - BA$

We then write the *commutator* product of the angular momentum **bivector operators**

(6.284) $[\mathrm{L}_1,\mathrm{L}_2] = 2[\mathrm{L}_1\times\mathrm{L}_2] = \mathrm{L}_1\mathrm{L}_2 - \mathrm{L}_2\mathrm{L}_1 = \hbar\mathrm{L}_3,$ or by orthogonality just $\mathrm{L}_1\mathrm{L}_2 = \hbar\frac{1}{2}\mathrm{L}_3.$

The reason for this is:

- First, for 1-vector operator commutation using (6.280) joined by ~(6.279)

(6.285) $[\mathrm{j}_2,\mathrm{j}_1] = \mathrm{j}_2\mathrm{j}_1 - \mathrm{j}_1\mathrm{j}_2 = \hbar^2\lambda_1\lambda_2\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 - \hbar^2\lambda_1\lambda_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = \hbar^2 2\lambda_1\lambda_2\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \hbar^2 2\lambda_1\lambda_2\boldsymbol{i}_3 \sim \hbar\mathrm{L}_3 = \hbar^2\lambda_3\boldsymbol{i}_3.$

To make this fit we demand $\boxed{\lambda_3 = 2\lambda_1\lambda_2}$. We note $\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3 \Rightarrow \mathrm{L}_3 = \boldsymbol{i}\mathrm{j}_3 = \hbar\lambda_3\boldsymbol{i}_3 = \hbar\lambda_3\boldsymbol{i}\boldsymbol{\sigma}_3,$ and compare with traditional QM (3.77) $[\hat{L}_1,\hat{L}_2] = i\hbar\hat{L}_3$, whereby we now write

(6.286) $[\mathrm{j}_2,\mathrm{j}_1] = \hbar\boldsymbol{i}\,\mathrm{j}_3 = \hbar\mathrm{L}_3.$

In a short form of angular momentum 1-vector multiplication, we just write

(6.287) $\mathrm{j}_2\mathrm{j}_1 = \hbar\boldsymbol{i}\frac{1}{2}\mathrm{j}_3 = \hbar\frac{1}{2}\mathrm{L}_3,$

This product of 1-vector operators is a bivector for the resulting angular momentum.[336]

- Then, for the dual bivector product commutator we write using (6.279) and the idea (6.263)

[335] These four real scalar operators $\lambda_\mu,\ \mu=0,1,2,3$ are a priori unknown variable ***quantities*** transcendental to our intuition.

[336] For intuition, the reader may benefit from Figure 5.5 and Figure 5.47.