Note the peculiar difference in angular ***direction*** orientation by the dextral definitions

1. from $\boldsymbol{\sigma}_3$ to $\mathbf{e}_1$ then to $\boldsymbol{\sigma}_1(\phi)$ all in the $\boldsymbol{i}_2(\phi)$ plane for $\odot_2$ (from the pole $\boldsymbol{\sigma}_3$), compared to the orientation.
2. first from $\mathbf{e}_2$ then further to $\boldsymbol{\sigma}_2(\phi)$ then to $\boldsymbol{\sigma}_3$ all in the $\boldsymbol{i}_1(\phi)$ plane for $\odot_1$ (towards the pole).

Considering the phase factor $\boldsymbol{i}_3 = e^{\boldsymbol{i}_3 \frac{1}{2}\pi} \Rightarrow \boldsymbol{i}_2 = \boldsymbol{i}_3\boldsymbol{i}_1$ to the development angle in the two cases that have the opposite orientations. The rotor oscillators $U_\psi := e^{\frac{1}{2}\boldsymbol{i}_2\psi}$ and $U_{\psi_1} := e^{\frac{1}{2}\boldsymbol{i}_1\psi_1}$ do not commute even for $\psi_1 = -\psi - \pi$, because they are synchronous perpendicular.
The co-frequency claim $\psi \sim \pm\omega t \sim \psi_1$ tell us $\psi_1 = \pm\psi + \theta$, with all phases $\forall\theta \in [0,2\pi[$ constant.
The main purpose of the two internal 1-vector ***directions*** $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ is for us free to define in our intuition the external ***direction*** $\boldsymbol{i}_3 := \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ and by that the external frame ***direction***

(6.253) $\mathbf{e}_3 = \boldsymbol{\sigma}_3 = -\boldsymbol{i}\,\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$.

This double cover with $U_2 := e^{\frac{1}{2}\boldsymbol{i}_2\psi}$ and $U_1 := e^{\frac{1}{2}\boldsymbol{i}_2\psi_1}$ for the development parameter ψ is due to the interconnectivity expressed in (6.123),(6.126),(6.130) and (6.145a)-(6.145f) for the even geometric algebra $\mathcal{G}_{0,2}$ of our $\mathfrak{Z}$-space ***entity*** $\Psi_{\phi,\psi}$. This double cover makes that we have both rotation oscillators $U_2^2 = e^{\boldsymbol{i}_2\psi}$ and $U_1^2 = e^{\boldsymbol{i}_1\psi_1}$ perpendicular to the $\boldsymbol{i}_3$ plane. These are driven in an oscillating around $\boldsymbol{\sigma}_3$ too, by $U_3^2 = e^{\boldsymbol{i}_3\phi}$ seen from the surroundings.
In all, we must consider the general unitary quantum uncertainty symmetry factor

(6.254) $\odot = \{\theta \to e^{\boldsymbol{i}\theta} \,|\, \forall\theta \in \mathbb{R}, \forall\boldsymbol{i} \in \mathfrak{Z}\}$,

a circular symmetry in ***all pqg***-2 ***directions*** in $\mathfrak{Z}$-space, a sphere symmetry concerning the physical foundation of the meaning with the locality of the even geometric algebra $\mathcal{G}_{0,2}(\mathbb{R})$.
The quaternion basis separates this in three fundamental orthogonal ***directions*** therefore we consider the three interconnected 1-spinors $\psi_1^{\frac{1}{2}} = \rho U_1$, $\psi_2^{\frac{1}{2}} = \rho U_2$ and $\psi_3^{\frac{1}{2}} = \varrho U_3$ which make a superposition to $\psi^{\frac{1}{2}} = \psi_3^{\frac{1}{2}} + \psi_2^{\frac{1}{2}} + \psi_1^{\frac{1}{2}} \leftrightarrows \hat{Q} = u_0 + u_3\boldsymbol{i}_3 + u_2\boldsymbol{i}_2 + u_1\boldsymbol{i}_1\ \psi^{\frac{1}{2}\dagger}$ that equals the quaternion form, that together with its own reverse $\psi^{\frac{1}{2}\dagger} \leftrightarrow \hat{Q}^\dagger$ describe a fully ***entity***. This is an alternative view to what we have seen, that this quaternion can be full described by two independent spinors, e.g. (6.169) from (6.145f): $\hat{Q} = Q_3 1 + Q_1\boldsymbol{i}_2$.
This is the closest we here will come to an illustration for the intuition of the simplest $\mathfrak{Z}$-space structure below the transcendental ***quantum*** barrier of a fundamental ***entity*** Ψ_3.

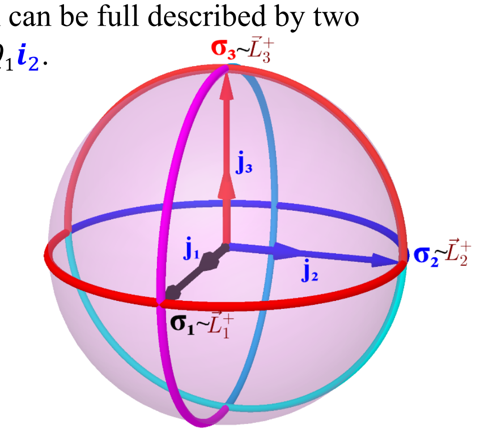

Figure 6.21 Three *internal* ***directions*** displayed as circle oscillators from the even algebra of rotors that is transversal orthogonal to the 1-vector ***directions*** $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$. Only *one external* $\vec{L}_k^+$ possible. The possible *internal* variable operator for the angular momentum components $\mathbf{j}_k$ are illustrated as 1-vectors operators. The object illustration presumes $|\mathbf{j}_k| \sim \frac{1}{2}$.

6.4.9.5. External and Internal *Directions* of an *Entity* in $\mathfrak{Z}$ space

When we speak of an object ***direction*** for an ***entity***, we mean an *external* ***quality***. For one specific ***entity***, we presume only one external ***direction***. In $\mathfrak{Z}$-space this can be expressed as a 1-vector joint by its dual bivector.
The information to the external can be given by a subton whose angular momentum gives the ***direction***, e.g., $\boldsymbol{\sigma}_3 \sim \vec{L}_3^+$.
Anyway, *internal* in the locality of ***entity*** Ψ_3 we are forced to look at the extension of three spatial ***directions***.
The best way to do this is by a geometric algebra basis as expressed in (6.119), which has an interconnectivity structure simply expressed in (6.126) and more detailed in section 6.4.3 with the even algebra quaternions.
The foundation of this problem is intuited in Figure 6.21.

6.4.9.6. Complementarity

Complementary to the angular circular wave oscillation we will now look at the angular momentum ***direction*** operators.

6.5. The Angular Momentum in $\mathfrak{Z}$ Space

6.5.1. One Quantum of Angular Momentum

From the review in § 6.4.9.1, we conclude the fundamental ***quantum*** is the excitation of a circular oscillation with an autonomous angular momentum of the ***quantity one*** ($\hbar$=1).
The fundamental ***quality*** of this is a ***primary quality of second grade*** (***pqg***-2), as a unit plane pseudoscalar $\boldsymbol{i}$ in an oscillation plane or a unit bivector in a space of higher ***grades***, e.g., $\mathfrak{Z}$-space.
This principle was first expressed in Kepler's second law as constant chronometric angular areas.
The classical view is that angular momentum is represented by a ***pqg***-1-vector ***direction*** as $\vec{L}_3^+$ (6.227). Here with ***quantum one*** $|\vec{L}_3^+| = \hbar$=1. To intuit this in a standard orthogonal 1-vector basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ for an $\mathfrak{Z}$-space ***entity*** we choose $\boldsymbol{\sigma}_3 = \vec{L}_3^+$. Dual to this in the even $\mathcal{G}_{3,0}^+ \sim \mathcal{G}_{0,2}$ Geometric Algebra this is expressed as (6.235) $\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3 = \boldsymbol{i}\vec{L}_3^+$, that's a transversal unit bivector defined by $\boldsymbol{i}_3 := \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$, as the plane pseudoscalar unit for an angular momentum ***quantum***.

6.5.1.1. 1-vector Angular Momentum

From classical physics, we have that the angular momentum as a 1-vector can be written as (3.62)

(6.255) $\vec{L} = \vec{r} \times \vec{p} = \mathbf{q} \times \mathbf{p}$

Where the position 1-vector from a center by position coordinate in the frame $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ is

(6.256) $\vec{r} = \mathbf{q} = q_1\boldsymbol{\sigma}_1 + q_2\boldsymbol{\sigma}_2 + q_3\boldsymbol{\sigma}_3$,

and the ***directional*** momentum 1-vector ***quantity*** with components along $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ is

(6.257) $\vec{p} = \mathbf{p} = p_1\boldsymbol{\sigma}_1 + p_2\boldsymbol{\sigma}_2 + p_3\boldsymbol{\sigma}_3$,

where both the coordinates $q_k \in \mathbb{R}$ and components $p_k \in \mathbb{R}$ for k=1,2,3 are real field scalars.
Here by the angular momentum 1-vector can be expressed from this basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$

(6.258) $\vec{L} = \lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2 + \lambda_3\boldsymbol{\sigma}_3$,

Where the component factors are the real determining scalars[331]

(6.259) $\lambda_1 = (q_2p_3 - q_3p_2) \in \mathbb{R}$,
(6.260) $\lambda_2 = (q_3p_1 - q_1p_3) \in \mathbb{R}$,
(6.261) $\lambda_3 = (q_1p_2 - q_2p_1) \in \mathbb{R}$.

6.5.1.2. The Bivector Angular Momentum

The angular momentum is expressed as a transversal bivector by the *outer product* (5.58)

(6.262) $\mathbf{L} = \boldsymbol{i}\vec{L} = \boldsymbol{i}(\mathbf{q}\times\mathbf{p}) = -\mathbf{q}\wedge\mathbf{p} = \mathbf{p}\wedge\mathbf{q} = \frac{1}{2}(\mathbf{pq} - \mathbf{qp}) \quad = \boldsymbol{i}(\vec{r}\times\vec{p}) = -\vec{r}\wedge\vec{p} = -\frac{1}{2}(\vec{r}\vec{p} - \vec{p}\vec{r})$

$$= \tfrac{1}{2}\big((p_1\boldsymbol{\sigma}_1 + p_2\boldsymbol{\sigma}_2 + p_3\boldsymbol{\sigma}_3)(q_1\boldsymbol{\sigma}_1 + q_2\boldsymbol{\sigma}_2 + q_3\boldsymbol{\sigma}_3) - (q_1\boldsymbol{\sigma}_1 + q_2\boldsymbol{\sigma}_2 + q_3\boldsymbol{\sigma}_3)(p_1\boldsymbol{\sigma}_1 + p_2\boldsymbol{\sigma}_2 + p_3\boldsymbol{\sigma}_3)\big)$$
$$= (q_2p_3 - q_3p_2)\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2 + (q_3p_1 - q_1p_3)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 + (q_1p_2 - q_2p_1)\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1.$$

Hence due to (6.123)

(6.263) $\mathbf{L} = \lambda_1\boldsymbol{i}_1 + \lambda_2\boldsymbol{i}_2 + \lambda_3\boldsymbol{i}_3 = \boldsymbol{i}(\lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2 + \lambda_3\boldsymbol{\sigma}_3) = \boldsymbol{i}\vec{L} = \mathbf{p}\wedge\mathbf{q} = -\mathbf{q}\wedge\mathbf{p} = \vec{p}\wedge\vec{r} = -\vec{r}\wedge\vec{p}$.

In ***quantum mechanics*** (QM) $\hbar$=1 we express the momentum operator (3.3), (3.68) as

(6.264) $\hat{p}_j = -i\hbar\frac{\partial}{\partial q_j}$, with j=1,2,3,

for the components, and further expressed as a 1-vector momentum operator

(6.265) $\hat{\vec{p}} = -i\hbar\boldsymbol{\nabla}$, where $\boldsymbol{\nabla} = \sum_j \frac{\partial}{\partial q_j} = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3}$

express the traditional 1-vector gradient operator $\boldsymbol{\nabla}$ (summed from three ***directions***)

[331] The reader may refer to chapter I. section 3.1.7, 3.1.8 for a classical view, and 3.1.9, 3.2 for a quantum mechanical view.