6.4.8. The Rotated Direction in $\mathfrak{Z}$-space

We look at rotations from the group of versor quaternions[326] described from a basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ (6.134) in the simple real linear unitary form (6.136)

(6.204) $U = u_0 + u_1\boldsymbol{i}_1 + u_2\boldsymbol{i}_2 + u_3\boldsymbol{i}_3 \in \mathbb{H}$, where $UU^\dagger = u_0^2+u_1^2+u_2^2+u_3^2 = 1$, and $\forall u_\kappa \in \mathbb{R}$.

We start with the simplified[327] 1-rotor versor elements of the angular form with an angle φ

(6.205) $U_\varphi = 1\cos½\varphi + \boldsymbol{i}_\varphi \sin½\varphi = e^{+\boldsymbol{i}_\varphi ½\varphi}$, as multivector $= U = \mathbf{vu} = \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\wedge\mathbf{u} = e^{+½\boldsymbol{\varphi}}$,

where $\boldsymbol{i}_\varphi$ is the bivector unit for the rotation plane ***direction*** of this circular angular form.
To get a full regular rotation $\underline{\mathcal{R}}(\mathbf{x}) = \underline{\mathcal{R}}\mathbf{x} = U\mathbf{x}U^\dagger$ of ***directions*** in $\mathfrak{Z}$-space we combine two $U(1)$ plane circular 1-rotors to a 2-rotor $U = U_\phi U_\theta$ and by that achieve two angular independent parameters θ and ϕ arguments, e.g. unit *spherical coordinates* $(1, \theta, \varphi)$ for S^2.
In our practice, we prefer to represent this independency by two orthogonal angular bivectors

(6.206) $\boldsymbol{\theta} = \theta\boldsymbol{i}_2$ and $\boldsymbol{\varphi} = \varphi\boldsymbol{i}_3$

where the two perpendicular plane ***directions*** are given by the two chosen orthogonal basis bivectors $\boldsymbol{i}_2$ and $\boldsymbol{i}_3$ that autonomous implies the third basis bivector ***direction*** $\boldsymbol{i}_1 = \boldsymbol{i}_2\boldsymbol{i}_3$.
A scalar has no ***direction***, so its unit is a pure 1 and the versor basis is just $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$.
Whence we write the two $U(1)$ 1-rotors as $\boxed{U_\theta := e^{½\boldsymbol{i}_2\theta}} = e^{+½\boldsymbol{\theta}}$ and $\boxed{U_\varphi = e^{½\boldsymbol{i}_3\varphi}} = e^{+½\boldsymbol{\varphi}}$.
We combine these to a new $SU(2)$ isomorph 2-*rotor* by multiplication

(6.207) $U = U_\varphi U_\theta = e^{+½\boldsymbol{\varphi}}e^{+½\boldsymbol{\theta}} = e^{+½\boldsymbol{i}_3\varphi}e^{+½\boldsymbol{i}_2\theta}$, and its reverse $U^\dagger = U_\theta^\dagger U_\varphi^\dagger = e^{-½\boldsymbol{i}_2\theta}e^{-½\boldsymbol{i}_3\varphi}$.

6.4.8.2. Rotation of a Chosen Direction in $\mathfrak{Z}$ Space

In the Cartesian tradition the ***directions*** in $\mathfrak{Z}$-space are given by an orthonormal 1-vector dextral basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$. By this we select the chosen ***direction*** as $\boldsymbol{\sigma}_3 = -\boldsymbol{\imath}\,\boldsymbol{i}_3$.
For the regular rotation of a chosen ***direction*** $\boldsymbol{\sigma}_3$ we use what we called *the canonical form for any orthogonal transformation* (5.193) and (6.70)

(6.208) $\mathbf{n} = \underline{\mathcal{R}}\boldsymbol{\sigma}_3 = U\boldsymbol{\sigma}_3U^\dagger = U_\varphi U_\theta\boldsymbol{\sigma}_3U_\theta^\dagger U_\varphi^\dagger = e^{+½\boldsymbol{i}_3\varphi}e^{+½\boldsymbol{i}_2\theta}\boldsymbol{\sigma}_3e^{-½\boldsymbol{i}_2\theta}e^{-½\boldsymbol{i}_3\varphi}$

Separating the operating factors to achieve the form (6.204)

(6.209) $U = U_\varphi U_\theta = (\cos½\varphi + \boldsymbol{i}_3\sin½\varphi)(\cos½\theta + \boldsymbol{i}_2\sin½\theta)$

$= \cos½\varphi\cos½\theta - \boldsymbol{i}_1\sin½\varphi\sin½\theta + \boldsymbol{i}_2\cos½\varphi\sin½\theta + \boldsymbol{i}_3\sin½\varphi\cos½\theta$,

and the reverse

(6.210) $U^\dagger = U_\theta^\dagger U_\varphi^\dagger = (\cos½\theta - \boldsymbol{i}_2\sin½\theta)(\cos½\varphi - \boldsymbol{i}_3\sin½\varphi)$

$= \cos½\varphi\cos½\theta + \boldsymbol{i}_1\sin½\varphi\sin½\theta - \boldsymbol{i}_2\cos½\varphi\sin½\theta - \boldsymbol{i}_3\sin½\varphi\cos½\theta$.

Setting the scalar coefficients in front, and then left multiply the reversed by $\boldsymbol{\sigma}_3$

(6.211) $U = \cos½\varphi\cos½\theta - \sin½\varphi\sin½\theta\,\boldsymbol{i}_1 + \cos½\varphi\sin½\theta\,\boldsymbol{i}_2 + \sin½\varphi\cos½\theta\,\boldsymbol{i}_3$,

(6.212) $\boldsymbol{\sigma}_3U^\dagger = \cos½\varphi\cos½\theta\,\boldsymbol{\sigma}_3 + \sin½\varphi\sin½\theta\,\boldsymbol{\sigma}_2 + \cos½\varphi\sin½\theta\,\boldsymbol{\sigma}_1 - \sin½\varphi\cos½\theta\,\boldsymbol{\imath}$,

because: $\boldsymbol{\sigma}_3$, $\boldsymbol{\sigma}_3\boldsymbol{i}_1 = \boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3\boldsymbol{i}_2 = -\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_3\boldsymbol{i}_3 = \boldsymbol{\imath}$.

From

1,1:1:	$\boldsymbol{\sigma}_3$,	2,2:2:	$\boldsymbol{i}_1\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_3$,	3,3:3:	$\boldsymbol{i}_2\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_3$,	4,4:4:	$-\boldsymbol{i}_3\boldsymbol{\imath} = \boldsymbol{\sigma}_3$,	$\boldsymbol{\sigma}_3$
2,1:1:	$-\boldsymbol{i}_1\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_2$,	1,2:2:	$\boldsymbol{\sigma}_2$,	4,3:3:	$\boldsymbol{i}_3\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$,	3,4:4:	$-\boldsymbol{i}_2\boldsymbol{\imath} = \boldsymbol{\sigma}_2$,	$\boldsymbol{\sigma}_2$
3,1:1:	$\boldsymbol{i}_2\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1$,	4,2:3:	$-\boldsymbol{i}_3\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1$,	1,3:2:	$\boldsymbol{\sigma}_1$,	2,4:4:	$-\boldsymbol{i}_1\boldsymbol{\imath} = \boldsymbol{\sigma}_1$,	$\boldsymbol{\sigma}_1$
4,1:1:	$\boldsymbol{i}_3\boldsymbol{\sigma}_3 = \boldsymbol{\imath}$,	3,2:3:	$\boldsymbol{i}_2\boldsymbol{\sigma}_2 = \boldsymbol{\imath}$,	2,3:4:	$\boldsymbol{i}_1\boldsymbol{\sigma}_1 = \boldsymbol{\imath}$,	1,4:2:	$\boldsymbol{\imath}$,	$\boldsymbol{\imath}$

we get by (6.211) U operating on the product (6.212) $\boldsymbol{\sigma}_3U^\dagger$ the specific ***direction*** from (6.208)

[326] This is synonymous with the $SO(3)$ group from the Spin(3) Lie group isomorphic to the special unitary group $SU(2)$.
[327] With a simplified rotor, we understand a rotor that only exists along its own plane and therefore belongs to the $U(1)$ group.

$\mathbf{n} = U_\varphi U_\theta\boldsymbol{\sigma}_3U_\theta^\dagger U_\varphi^\dagger = U\boldsymbol{\sigma}_3U^\dagger =$

(6.213) $+(\cos½\varphi\cos½\theta)^2\boldsymbol{\sigma}_3 - (\sin½\varphi\sin½\theta)^2\boldsymbol{\sigma}_3 - (\cos½\varphi\sin½\theta)^2\boldsymbol{\sigma}_3 + (\sin½\varphi\cos½\theta)^2\boldsymbol{\sigma}_3$
$+\cos½\varphi\sin½\varphi\cos½\theta\sin½\theta\,(\boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_2)$
$+(\cos½\varphi)^2\cos½\theta\sin½\theta\,(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_1) - (\sin½\varphi)^2\sin½\theta\cos½\theta\,(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_1)$
$+\cos½\varphi\sin½\varphi\,(\cos½\theta)^2(\boldsymbol{\imath} - \boldsymbol{\imath}) - \sin½\varphi\cos½\varphi\,(\sin½\theta)^2(\boldsymbol{\imath} - \boldsymbol{\imath})$

The last pseudoscalar terms vanish, and we write out its 1-vector component ***directions***

(6.214) $n_3\boldsymbol{\sigma}_3 = \left((\cos½\varphi)^2 + (\sin½\varphi)^2\right)\left((\cos½\theta)^2 - (\sin½\theta)^2\right)\boldsymbol{\sigma}_3 = \cos\theta\,\boldsymbol{\sigma}_3$,

(6.215) $n_2\boldsymbol{\sigma}_2 = 2\cos½\varphi\sin½\varphi\,2\cos½\theta\sin½\theta\,\boldsymbol{\sigma}_2 = \sin\varphi\sin\theta\,\boldsymbol{\sigma}_2$,

(6.216) $n_1\boldsymbol{\sigma}_1 = 2\left((\cos½\varphi)^2 - (\sin½\varphi)^2\right)\cos½\theta\sin½\theta\,\boldsymbol{\sigma}_1 = \cos\varphi\sin\theta\,\boldsymbol{\sigma}_1$,

from which the unit 1-vector ***direction*** from (6.140) is

(6.217) $\mathbf{n} = U\boldsymbol{\sigma}_3U^\dagger = n_1\boldsymbol{\sigma}_1 + n_2\boldsymbol{\sigma}_2 + n_3\boldsymbol{\sigma}_3 = \sin(\theta)\left(\cos(\varphi)\,\boldsymbol{\sigma}_1 + \sin(\varphi)\,\boldsymbol{\sigma}_2\right) + \cos(\theta)\boldsymbol{\sigma}_3 \in (V_3, \mathbb{R})$, $|\mathbf{n}| = 1$,

whose Cartesian coordinates are expressed by its unit *spherical coordinates* $(1, \theta, \varphi)$

(6.218) $n_1 = \cos\varphi\sin\theta$, $n_2 = \sin\varphi\sin\theta$, $n_3 = \cos\theta$ } with pole angle θ, and azimuthal angle φ

This is the *spherical* map $\underline{\mathcal{R}}_{(\theta,\varphi)}: \boldsymbol{\sigma}_3 \longrightarrow \mathbf{n}$, which specifies a ***direction*** relative to the polar ***direction***.
When it comes to the versor quaternion plane ***direction*** we can translate from (6.217) to the dual space by using (6.142)

(6.219) $\boldsymbol{i}_\mathbf{n} = \boldsymbol{\imath}\mathbf{n} = n_1\boldsymbol{i}_1 + n_2\boldsymbol{i}_2 + n_3\boldsymbol{i}_3$, where $\boldsymbol{i}_\mathbf{n}{}^2 = -1$ and $n_1^2+n_2^2+n_3^2 = 1$

This is the *spherical* rotation map $\underline{\mathcal{R}}_{(\theta,\varphi)}: \boldsymbol{i}_3 \longrightarrow \boldsymbol{i}_\mathbf{n}$.
By this, we get a resulting transversal bivector ***direction*** $\boldsymbol{i}_\mathbf{n} = \boldsymbol{\imath}\mathbf{n}$ of a simplified circular 1-rotor

(6.220) $U_{\varphi_\mathbf{n}} = e^{\boldsymbol{\imath}\mathbf{n}½\varphi_\mathbf{n}} = e^{+\boldsymbol{i}_\mathbf{n}½\varphi_\mathbf{n}} = 1\cos½\varphi_\mathbf{n} + \boldsymbol{i}_\mathbf{n}\sin½\varphi_\mathbf{n}$

Now we will imagine that the angular parameter $\varphi_\mathbf{n} = \omega t$ is a development parameter in one oscillation, which has its own autonomous 1-vector ***direction*** $\mathbf{n}$.
On top of this, we will also imagine that the ***direction*** $\mathbf{n}$ can oscillate too. Then the ***direction*** of the circular plane $\boldsymbol{i}_\mathbf{n}$ is oscillating too, then (6.220) is not as simple as it looks.
E.g., an oscillation in the $\boldsymbol{i}_3$ plane by the function $e^{\boldsymbol{i}_3\varphi_3} = e^{\boldsymbol{i}_3\omega_3t}$ and along $\boldsymbol{i}_2$ by $e^{\boldsymbol{i}_2\varphi_2} = e^{\boldsymbol{i}_2\omega_2t}$

(6.221) $\boldsymbol{i}_\mathbf{n}(\phi_3) = e^{\boldsymbol{i}_3\varphi_3}\boldsymbol{i}_{\mathbf{n}_0}$ and $\boldsymbol{i}_\mathbf{n}(\varphi_2) = e^{\boldsymbol{i}_2\varphi_2}\boldsymbol{i}_{\mathbf{n}_0}$.

The reader may in thought consider the impact, that a merge of these angular development parameters φ_3, φ_2 in the two governed 1-rotors, and imagine the form of $\boldsymbol{i}_\mathbf{n}(\varphi_3, \varphi_2) = \boldsymbol{\imath}\mathbf{n}(\varphi_3, \varphi_2)$ for the resulting rotating, rotation plane.

A comment to the inverse of (6.208):

(6.222) $\boldsymbol{\sigma}_3 = \underline{\mathcal{R}}^{-1}(\mathbf{n}) = U^\dagger\mathbf{n}U = U_\phi^\dagger U_\theta^\dagger\mathbf{n}U_\theta U_\phi = e^{-½\boldsymbol{i}_3\phi}e^{-½\boldsymbol{i}_2\theta}\mathbf{n}\,e^{+½\boldsymbol{i}_2\theta}e^{+½\boldsymbol{i}_3\phi}$.

It just gives the same as reversing both angular parameters,

(6.223) $-n_1 = \cos{-\varphi}\sin{-\theta}$, $n_2 = \sin{-\varphi}\sin{-\theta}$, $n_3 = \cos{-\theta}$

Therefore, not a parity inversion but the double reversion is just a reflection in one plane $\boldsymbol{i}_1$

(6.224) $\mathbf{n}' = \underline{\mathcal{R}}^{-1}\boldsymbol{\sigma}_3 = U^\dagger\boldsymbol{\sigma}_3U = U_\varphi^\dagger U_\theta^\dagger\boldsymbol{\sigma}_3U_\theta U_\varphi = -n_1\boldsymbol{\sigma}_1 + n_2\boldsymbol{\sigma}_2 + n_3\boldsymbol{\sigma}_3$.