Now we will compare the Euler angles of the rotor (6.187) with the versor quaternion expression (6.136) $U = \hat{Q} = u_0 + u_1\boldsymbol{i}_1 + u_2\boldsymbol{i}_2 + u_3\boldsymbol{i}_3 \in \mathbb{H}$, therefor we rewrite (6.187)

(6.194)
$$\begin{aligned} U &= U_\phi U_\theta U_\psi = e^{½\boldsymbol{i}_3\phi} e^{½\boldsymbol{i}_1\theta} e^{½\boldsymbol{i}_3\psi} \\ &= (\cos ½\phi + \boldsymbol{i}_3 \sin ½\phi)(\cos ½\theta + \boldsymbol{i}_1 \sin ½\theta)(\cos ½\psi + \boldsymbol{i}_3 \sin ½\psi) \\ &= \cos ½\theta\,(\cos ½\phi \cos ½\psi - \sin ½\phi \sin ½\psi) \\ &\quad + \sin ½\theta\,(\cos ½\phi \cos ½\psi - \sin ½\phi \sin ½\psi)\, \boldsymbol{i}_1 \\ &\quad + \sin ½\theta\,(\sin ½\phi \cos ½\psi - \cos ½\phi \sin ½\psi)\, \boldsymbol{i}_2 \\ &\quad + \cos ½\theta\,(\sin ½\phi \cos ½\psi + \cos ½\phi \sin ½\psi)\, \boldsymbol{i}_3 \end{aligned}$$

We have the unitary versor quaternion coordinates:

(6.195)
$$\begin{aligned} u_0 &= \cos ½\theta \, \cos\big(½(\phi+\psi)\big) \\ u_1 &= \sin ½\theta \, \cos\big(½(\phi+\psi)\big) \\ u_2 &= \sin ½\theta \, \sin\big(½(\phi-\psi)\big) \\ u_3 &= \cos ½\theta \, \sin\big(½(\phi+\psi)\big) \end{aligned}$$

By this, we have specified the versor quaternion by the Euler angles, e.g., for case (6.145e)

(6.196) $$\hat{Q} = u_0 + u_3\boldsymbol{i}_3 + u_1\boldsymbol{i}_1 + u_2\boldsymbol{i}_2 = (u_0 + u_3\boldsymbol{i}_3) + (u_1 + u_2\boldsymbol{i}_3)\boldsymbol{i}_1.$$

We will not study these practice rotations in more detail for 3-space here but mention that versor quaternions are used a lot in computer graphics, robotics, flight, and space satellite management.

6.4.6.2. The Other Euler Angle Sequence

The same Euler angles parameters as used in (6.187) where instead of rotating in the ***entity*** Ψ reference system $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ with duals $\boldsymbol{i}_k = \boldsymbol{\imath}\boldsymbol{\sigma}_k$ is rotating the reference frame in two sequences:

a. First, we choose the reference system as (6.179) with the primary rotation axis $\boldsymbol{\sigma}_3$ in duality with the plane unit bivector $\boldsymbol{i}_3 = \boldsymbol{\imath}\boldsymbol{\sigma}_3$ ***direction***. Around in this defining an arbitrary rotor $\boxed{U_\phi = e^{½\boldsymbol{i}_3\phi}}$ and calling its angle ϕ for the outer Euler angle for the frame rotation to a note ***direction*** $\mathbf{n} = \mathbf{e}_1''' = U_\phi^\dagger \boldsymbol{\sigma}_1 U_\phi$ as a new intermediary reference frame $\{\mathbf{n}, \mathbf{e}_2''', \boldsymbol{\sigma}_3\}$.

b. Second, by the newly created plane ***direction*** $\boldsymbol{i}_\mathbf{n} = \boldsymbol{\imath}\mathbf{n}$ dual to the intermediate node $\mathbf{n}$. In that we define an arbitrary rotor $\boxed{U_\theta := e^{½\boldsymbol{\imath}\mathbf{n}\theta}}$ and call its angle θ for the tilting Euler angle of frame tilt rotation in this ***pqg*-2*direction*** $\boldsymbol{\imath}\mathbf{n}$ plane, θ is taken from the $\boldsymbol{\sigma}_3$ axis tilting ***direction*** $\boldsymbol{i}_3$ plane to the ***direction*** $\boldsymbol{\imath}\mathbf{e}_3 = \mathbf{e}_2'''\mathbf{n}$, resulting in a new frame $\{\mathbf{n}, \mathbf{e}_2'''', \mathbf{e}_3, \ \}$ dual to $\{\boldsymbol{i}_\mathbf{n}, \mathbf{e}_3\mathbf{n}, \boldsymbol{\imath}\mathbf{e}_3\}$.

c. Third, in this new frame we have the rotation axis $\mathbf{e}_3$ with the dual transversal plane $\boldsymbol{\imath}\mathbf{e}_3$ around which symmetry ***direction*** we define the third arbitrary rotor $\boxed{U_\psi := e^{½\boldsymbol{\imath}\mathbf{e}_3\psi}}$ and call its angle ψ for the inner Euler angle, as the rotation in the ***entity*** Ψ ***direction*** $\boldsymbol{\imath}\mathbf{e}_3$.

In this frame scenario, we combine this sequence of the three 1-rotors (as used by Doran & Lasenby [18]p.51) to one unitary rotor operator U, that can perform the total rotation

(6.197) $$U = U_\psi U_\theta U_\phi = e^{½\boldsymbol{\imath}\mathbf{e}_3\psi} e^{½\boldsymbol{\imath}\mathbf{n}\theta} e^{½\boldsymbol{i}_3\phi}.$$

This resulting rotor is just the same as (6.187) used in (6.193) and (6.192)

(6.198) $$\mathbf{e}_k = U^\dagger \boldsymbol{\sigma}_k U \quad \text{and} \quad \Psi_3 = U\Psi U^\dagger.$$

As the reader may know there are other systems to interpret Euler angles relative to the axis. (The structure in (6.145a)-(6.145f) indicates some of the possibilities of rotation planes.) Anyway, in robotics, the axis is distributed and cannot necessarily be traded as one locality in a 3-space.

6.4.7. The Transversal Bivector Idea Dual to a 1-vecetor Foundation for Rotations

Before we proceed, we have to be precise that a transversal bivector $\mathbf{B} = \mathbf{b}\boldsymbol{\imath} = \boldsymbol{\imath}\mathbf{b}$ is dual perpendicular orthogonal to its generating 1-vector $\mathbf{b} = \beta_1\boldsymbol{\sigma}_1 + \beta_2\boldsymbol{\sigma}_2 + \beta_3\boldsymbol{\sigma}_3$ and possesses the same coordinates to its dual basis, that is $\mathbf{B} = \beta_1\boldsymbol{i}_1, +\beta_2\boldsymbol{i}_2, +\beta_3\boldsymbol{i}_3 = \beta_1\boldsymbol{\imath}\boldsymbol{\sigma}_1 + \beta_2\boldsymbol{\imath}\boldsymbol{\sigma}_2 + \beta_3\boldsymbol{\imath}\boldsymbol{\sigma}_3$.

6.4.7.1. The two Orthogonal Rotors as Generators for a Local Entity

Now we start with the idea of the Euler angles from(6.187)-(6.196) for the rotors. We choose for the second 1-rotor to rotate perpendicular by the second Euler angle $\boxed{\theta = -\pi/2}$ and achieve

(6.199) $$\boxed{U_\theta := e^{½\boldsymbol{i}_1\theta}} \Rightarrow U_{-½\pi} = e^{-\boldsymbol{i}_1\pi/4} = \sqrt{½}(1 - \boldsymbol{i}_1) = \cos(-\pi/4) + \boldsymbol{i}_1 \sin(-\pi/4)$$

Where we have $\cos(-\pi/4) = \sqrt{½}$ and $\sin(-\pi/4) = -\sqrt{½}$.
This rotates the rotation axis $\boldsymbol{\sigma}_3$ over in the rotation axis $\boldsymbol{\sigma}_2$.

(6.200) $$U_{-½\pi}\boldsymbol{\sigma}_3 U^\dagger_{-½\pi} = \sqrt{½}(1-\boldsymbol{i}_1)\boldsymbol{\sigma}_3(1+\boldsymbol{i}_1)\sqrt{½} = ½(1 + \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_3(1 + \boldsymbol{\sigma}_3\boldsymbol{\sigma}_2) = \boldsymbol{\sigma}_2$$

We tilt the 1-rotor $\boxed{U_\psi := e^{½\boldsymbol{i}_3\psi}}$ to be orthogonal to 1-rotor $\boxed{U_\phi := e^{½\boldsymbol{i}_3\phi}}$

$$U_\psi \longrightarrow U_{-½\pi} U_\psi U^\dagger_{-½\pi} = U_\psi = e^{½\boldsymbol{\imath}\boldsymbol{\sigma}_2\psi} = e^{½\boldsymbol{i}_2\psi}$$

and have a set of two orthogonal 1-rotors $\boxed{U_\phi := e^{½\boldsymbol{i}_3\phi} \text{ and } U_\psi := e^{½\boldsymbol{i}_2\psi}}$

It is very important to note that these angular ***quantities*** ϕ and ψ are not additive commutative in that their ***qualities*** have different ***directions*** $\boldsymbol{i}_3 \perp \boldsymbol{i}_2$, (refer to § 6.3.5.2 for multiplication). Anyway, we use it apparently as such in calculating the versor $\hat{Q}_{\phi\perp\psi}$ coordinates from (6.195)

(6.201)
$$\begin{aligned} u_0 &= +\sqrt{½}\cos\big(½(\phi+\psi)\big) \\ u_1 &= -\sqrt{½}\cos\big(½(\phi+\psi)\big) \\ u_2 &= -\sqrt{½}\sin\big(½(\phi-\psi)\big) \\ u_3 &= +\sqrt{½}\sin\big(½(\phi+\psi)\big) \end{aligned} \qquad \hat{Q} = u_0 + u_1\boldsymbol{i}_1 + u_2\boldsymbol{i}_2 + u_3\boldsymbol{i}_3$$

where the trigonometric functions express this nonlinearity.
Therefore, we instead will focus on the resulting 2-rotor of these two orthogonal 1-rotors

(6.202) $$\Psi_3 \longrightarrow \boxed{U = U_{\phi,\psi} := U_\phi U_\psi = e^{½\boldsymbol{i}_3\phi} e^{½\boldsymbol{i}_2\psi}} \sim e^{½B^{\circlearrowleft}}$$ ↺ OBS! [325]

Two plane ***directions*** generated from $\boldsymbol{i}_3$ and $\boldsymbol{i}_2$ are sufficient to describe the total ***direction*** of a physical ***entity*** Ψ_3 in 3-space, as we know from the mutual interconnectivity (6.123), (6.126) the third plane ***direction*** is implicit given

(6.203) $$\{1,\ \boldsymbol{i}_1 := \boldsymbol{i}_2\boldsymbol{i}_3,\ \boldsymbol{i}_2 := \boldsymbol{i}_3\boldsymbol{i}_1,\ \boldsymbol{i}_3 := \boldsymbol{i}_1\boldsymbol{i}_2\}$$

as the basis for the even geometric algebra $\mathcal{G}_{0,2}$ for a locality in 3-space, where the ***primary qualities are of even grades*** (***pqg***-0 and ***pqg***-2). The three subject ***directions*** $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ will always intersect in just one geometric point of 3-space that will represent a centrum of the locality. The translation invariance of such three perpendicular plane objects will always ensure one centrum of locality for a physical ***entity*** Ψ_3 in 3-space. In § 6.1.3.4 we had first, that two inclining planes (E XI.De.6.) will intersect in a straight line. This line will intersect the third plane in just one autonomous point, it will be an origo for these three inclining planes.

[325] For this two-parameter rotor $\boxed{U_{\phi,\psi}} \not\approx U_\psi U_\phi$, for intuition you need to notice that the initial plane ***direction*** $\boldsymbol{i}_2$ as a subject is rotated by U_ϕ as $\boldsymbol{i}_2^{\circlearrowleft} = U_\phi \boldsymbol{i}_2 U_\phi^\dagger = e^{½\boldsymbol{i}_3\phi}\boldsymbol{i}_2 e^{-½\boldsymbol{i}_3\phi} = e^{\boldsymbol{i}_3\phi}\boldsymbol{i}_2$, so that the bivector $B^{\circlearrowleft} = ½\boldsymbol{i}_3\phi + ½\boldsymbol{i}_2^{\circlearrowleft}\psi$, gives the rotor plane ***direction*** for the resulting two-parameter rotor (6.202). Note the bivector linear combination by the two angular coordinates ϕ, ψ, where the ψ plane is rotated. The group of these $\boxed{U_{\phi,\psi}}$ is isomorphic to the $SU(2)$ group.
Further here, it is worth noting that $\boldsymbol{i}_1 = e^{-\boldsymbol{i}_3\pi/2}\boldsymbol{i}_2 = ½(1-\boldsymbol{i}_3)\,\boldsymbol{i}_2(1+\boldsymbol{i}_3)$, in that $U_{-½\pi} = e^{-\boldsymbol{i}_3\pi/4} = \sqrt{½}(1-\boldsymbol{i}_3)$, for the orthogonal bivector basis (6.203). – (For the intuition of the plane subjects look once again at Figure 6.1,t, etc.)

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