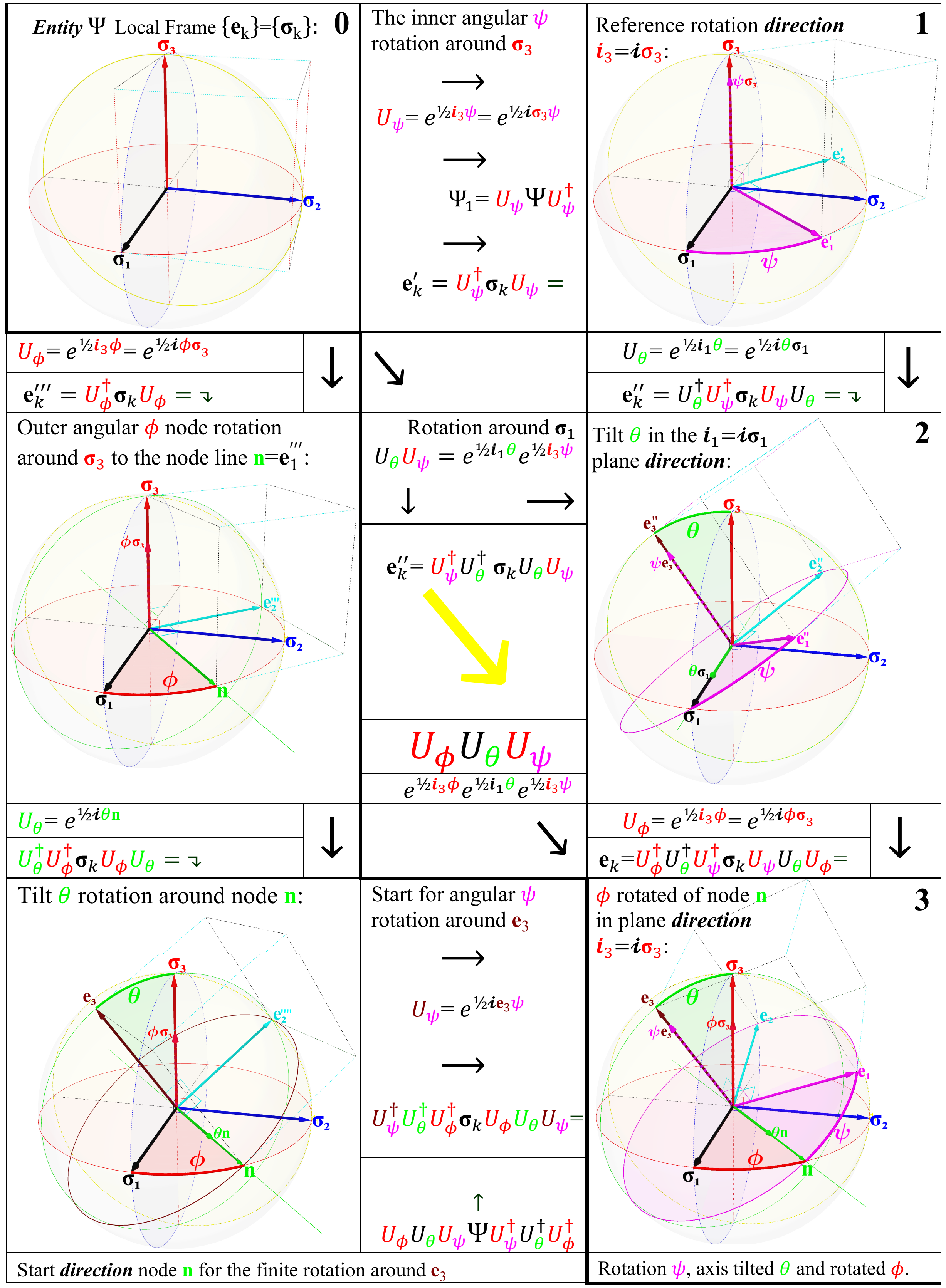

Figure 6.17 Overview of a simple rotation by the three Euler angles in an orthogonal local frame: 0. $\Psi\sim\{\sigma_1,\sigma_2,\sigma_3\}$.
1. The rotation angle ψ of the reference oscillation for the *entity* Ψ in the $i_3=i\sigma_3$ plane.
2. The tilting rotation angle θ in the $i_1=i\sigma_1$ plane. – 3. The start σ_1 rotated angle ϕ to node **n** in the $i_3=i\sigma_3$ plane.

We endow the situation of ***entity*** Ψ with the three ***principal directions*** given by the dextral orthonormal basis 1-vector $\{\sigma_1,\sigma_2,\sigma_3\}$ in duality with the transversal bivector basis $\{\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\}$.
We associate the angular rotations to these ***principal directions***.
How does this rotation of ***entity*** Ψ relate to the surroundings expressed as a frame of ***directions***?
An intuition of this as an orthonormal 1-vector frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\}$ is displayed in Figure 6.17.
We start with $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\}=\{\sigma_1,\sigma_2,\sigma_3\}$ in the first picture Figure 6.17,**0**.
For this regular rotation (6.184) we use the Euler angles as parameters: Displayed in Figure 6.17,

1. First, we choose a primary rotation symmetry ***direction*** of the local ***entity*** Ψ; call its unit bivector for $\boldsymbol{i}_3=\boldsymbol{i}\sigma_3$ given by what we call its basis 1-vector σ_3; around this, we define an arbitrary 1-rotor $\boxed{U_\psi:=e^{\frac12\boldsymbol{i}_3\psi}}$ and call its angle ψ for the inner Euler angle.
2. Second, we shall choose a start ***direction*** in this plane by a basis vector σ_1 in duality with its basis bivector $\boldsymbol{i}_1=\boldsymbol{i}\sigma_1$; in this plane define another arbitrary 1-rotor $\boxed{U_\theta:=e^{\frac12\boldsymbol{i}_1\theta}}$ and call its angle θ for the tilting Euler angle. – [321]
3. Third, around in the first symmetry ***direction*** $\boldsymbol{i}_3=\boldsymbol{i}\sigma_3$ we define the third arbitrary 1-rotor $\boxed{U_\phi=e^{\frac12\boldsymbol{i}_3\phi}}$ and call its angle ϕ for the outer Euler angle.

The theatre here is to combine these three 1-rotors to one unitary operator U, which performs the rotation of the frame at the ***entity*** Ψ. We make, this by the product[322] of the rotors:

(6.187) $$\hat{Q}\sim U=U_\phi U_\theta U_\psi=e^{\frac12\boldsymbol{i}_3\phi}e^{\frac12\boldsymbol{i}_1\theta}e^{\frac12\boldsymbol{i}_3\psi}.$$

Here U_ϕ,U_θ,U_ψ does not commutate as described in § 6.3.5 and shown in Figure 6.14, and we remember that the written operators as concept principal act from left on right, as an operator on the operand. (Refer to written functional principle $f\circ g=f(g)=fg\not\approx gf=g(f)=g\circ f$.)
The simple Hermitian conjugated operates from right to left position and are each rotor reversed

(6.188) $$\tilde{\hat{Q}}\sim U^\dagger=U_\psi^\dagger U_\theta^\dagger U_\phi^\dagger=e^{-\frac12\boldsymbol{i}_3\psi}e^{-\frac12\boldsymbol{i}_1\theta}e^{-\frac12\boldsymbol{i}_3\phi}$$

We look at the operation, we call the Euler angle extrinsic rotation in three sequential steps:

The rotation of frame $\{\sigma_1,\sigma_2,\sigma_3\}\longrightarrow\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\}$	Alternatively, rotation of the ***entity*** Ψ
First, we rotate around σ_3 in $\boldsymbol{i}_3$ by the inner rotor $U_\psi=e^{\frac12\boldsymbol{i}_3\psi}=e^{\frac12\boldsymbol{i}\psi\sigma_3}$:	
(6.189) $\mathbf{e}'_k=U_\psi^\dagger\sigma_k U_\psi$	$\Psi_1=U_\psi\Psi U_\psi^\dagger$
Second, we rotate around σ_1 in $\boldsymbol{i}_1$ by the tilt rotor $U_\theta=e^{\frac12\boldsymbol{i}_1\theta}=e^{\frac12\boldsymbol{i}\theta\sigma_1}$:	
(6.190) $\mathbf{e}''_k=U_\theta^\dagger U_\psi^\dagger\sigma_k U_\psi U_\theta$	$\Psi_2=U_\theta U_\psi\Psi U_\psi^\dagger U_\theta^\dagger$
Third, we rotate once again around σ_3 by the outer rotor $U_\phi=e^{\frac12\boldsymbol{i}_3\phi}=e^{\frac12\boldsymbol{i}\phi\sigma_3}$:	
(6.191) $\mathbf{e}_k=U_\phi^\dagger U_\theta^\dagger U_\psi^\dagger\sigma_k U_\psi U_\theta U_\phi=U^\dagger\sigma_k U$	$\Psi_3=U\Psi U^\dagger=U_\phi U_\theta U_\psi\Psi U_\psi^\dagger U_\theta^\dagger U_\phi^\dagger$

This three-step sequence of rotor rotations shown in Figure 6.17, 0→1→2→3 result in a total regular rotation, with an equivalence to the rotation (6.184) of the entity.

(6.192) $$\Psi_3=U\Psi U^\dagger\quad\sim\quad\mathcal{R}_{\mathbf{n}}\Psi=U_{\boldsymbol{i}\mathbf{n}}\Psi U_{\boldsymbol{i}\mathbf{n}}^\dagger=e^{\boldsymbol{i}\mathbf{n}\frac12\varphi}\Psi e^{-\boldsymbol{i}\mathbf{n}\frac12\varphi}=e^{\boldsymbol{i}\mathbf{n}\varphi}\Psi=(U_{\boldsymbol{i}\mathbf{n}})^2\Psi.$$

The total regular rotation of the frame is

(6.193) $$\mathbf{e}_k=U^\dagger\sigma_k U\quad\sim\quad\tilde{\mathcal{R}}_{\mathbf{n}}\sigma_k=U_{\boldsymbol{i}\mathbf{n}}^\dagger\sigma_k U_{\boldsymbol{i}\mathbf{n}}=e^{-\boldsymbol{i}\mathbf{n}\frac12\varphi}\sigma_k e^{\boldsymbol{i}\mathbf{n}\frac12\varphi}=e^{-\boldsymbol{i}\mathbf{n}\varphi}\sigma_k=(U_{\boldsymbol{i}\mathbf{n}}^\dagger)^2\sigma_k.$$

This review of Euler angles is inspired by Hestenes [10]p.289-292,[323] like [19]p.152.[324]

[321] The last basis vector σ_2 is implicitly given through $\boldsymbol{i}_2=\boldsymbol{i}_3\boldsymbol{i}_1$ by the two others, as $\sigma_2=-\boldsymbol{i}\boldsymbol{i}_2=\boldsymbol{i}\sigma_3\sigma_1$, and we do not use it for the Euler angles in this example. We only use it here for the intuition to define the planes $\boldsymbol{i}_3\equiv\sigma_2\sigma_1$ and $\boldsymbol{i}_1\equiv\sigma_3\sigma_2$.
[322] In the context of this book, this U is a multivector product. – In the quaternion picture, we call this versor $\hat{Q}$.
[323] Note Hestenes's canonical forms for linear operators. (The reader should study as much as possible of his book [10].)
[324] In this book, we don't follow the idea of a rigid body as in classical mechanics e.g., Goldstein [19], but only the form of 𝔍 space.