One e.g. (6.146) $\psi_{3+}^{½} \sim Q_3$ oscillating in the $\boldsymbol{i}_3$ plane as a chosen leading ***direction*** and perpendicular to this two oscillating 1-spinors $\psi_{2+}^{½} \sim Q_2^{\perp}$ and $\psi_{1+}^{½} \sim Q_1^{\perp}$, mutual perpendicular in every interconnected way as expressed in (6.151),(6.162),(6.163).

In all, for the three orthogonal ***directions*** we prescript three 1-spinors (dilated rotors)

(6.164) $$\psi_{3+}^{½} \sim \varrho_3 U_{\phi_3} = \varrho_3 e^{+½\boldsymbol{i}_3\phi_3} = \varrho_3(\cos ½\phi_3 + \boldsymbol{i}_3 \sin ½\phi_3),$$

(6.165) $$\psi_{2+}^{½} \sim \varrho_2 U_{\phi_2} = \varrho_2 e^{+½\boldsymbol{i}_2\phi_2} = \varrho_2(\cos ½\phi_2 + \boldsymbol{i}_2 \sin ½\phi_2) = \varrho_2\big(\sin ½(\phi_2+\pi) - \boldsymbol{i}_2 \cos ½(\phi_2+\pi)\big)$$

(6.166) $$\psi_{1+}^{½} \sim \varrho_1 U_{\phi_1} = \varrho_1 e^{+½\boldsymbol{i}_1\phi_1} = \varrho_1(\cos ½\phi_1 + \boldsymbol{i}_1 \sin ½\phi_1).$$

In total for one ***entity*** $\Psi_{½}$, we add these orthogonal wavefunctions $\psi^{½} = \psi_{3+}^{½} + \psi_{2+}^{½} + \psi_{1+}^{½} \sim \hat{Q}$.
Comparing these as multivectors with the versor quaternion (6.145) we see the sum of the scalars should match $u_0 = \varrho \cos ½\phi$ by

(6.167) $$u_0 = \varrho_3 \cos ½\phi_3 + \varrho_2 \cos ½\phi_2 + \varrho_1 \cos ½\phi_1.$$

What then with the bivector components?
It seems that all attempts to make a simple analytic expression of this will fail. Instead, we stick to the synthetic judgment that by the two-parameter definition (6.148)-(6.149) gives (6.145)

(6.168) $$\hat{Q} = u_0 + u_3\boldsymbol{i}_3 + u_1\boldsymbol{i}_1 + u_2\boldsymbol{i}_2 = \varrho(\cos ½\phi + \boldsymbol{i}_3 \sin ½\,\phi) + \rho(\boldsymbol{i}_2 \cos ½\psi + \boldsymbol{i}_1 \sin ½\psi)$$

(for intuition) $$= \varrho(\cos ½\phi + \boldsymbol{i}_3 \sin ½\,\phi) + \rho(\boldsymbol{i}_2 \sin ½(\psi+\pi) + \boldsymbol{i}_1 \sin ½\psi).$$

When we look at the wavefunction idea (6.164)-(6.166) as 1-spinor components in the even algebra $\mathcal{G}_{0,2}$ where their sum has to be a quaternion (6.131) $Q = q_0 + q_k\boldsymbol{i}_k$, and we can demand a normalization to (6.145) and obtain (6.168) autonomous govern by two independent angular parameters. In the picture of three fundamental orthogonal ***directions*** of an oscillating circle 1-spinors we accept a synchronisation $\phi \sim \psi \sim \phi_3 \sim \phi_2 \sim \phi_1 \sim \omega t$, all with all phases $\forall\theta \in [0,2\pi[$.
We call the versor (6.168) for a 2-rotor or a unit 2-spinor.

6.4.5. The Two Parameter Quaternion 2-Spinor

The traditional combination (5.157) is unusable for the two independent driving 1-spinors that just give a new 1-spinor, instead, we use the natural perpendicular orthogonality from e.g. (6.145f) for the driving ***direction*** $\boldsymbol{i}_3$ we have (6.146) and (6.147)

(6.169) $$Q_3 = (u_0 + u_3\boldsymbol{i}_3) \quad \text{and} \quad Q_1 = (u_2 + u_1\boldsymbol{i}_3), \qquad \text{with total} \quad U = \hat{Q} = Q_3 1 + Q_1\boldsymbol{i}_2$$

The extracted idea from formulation $\hat{Q} = Q_3 1 + Q_1\boldsymbol{i}_2$ is that we have two wavefunctions in superposition $\psi = \psi_{\text{transversal}} + \psi_{\text{orthogonal}} \in \mathbb{H}$. For the chosen ***direction*** $\boldsymbol{\sigma}_3$ one transversal $\psi_{\text{transversal}} \sim 1Q_3 \in \mathbb{H}$, and orthogonal to this wavefunction the other $\psi_{\text{orthogonal}} \sim Q_1\boldsymbol{i}_2 \in \mathbb{H}$.
In the tradition of quantum mechanics, there has been using complex numbers for these

(6.170) $$\alpha = u_0 + i\,u_3 \;\in \mathbb{C} \quad \leftrightarrow \quad Q_3 = (u_0 + u_3\boldsymbol{i}_3) \;\in \mathbb{H},$$

$\mathbb{T}$ is the circle group of $U(1)$.

(6.171) $$\beta = u_2 - i\,u_1 \;\in \mathbb{C} \quad \overset{\mathbb{T}}{\leftrightarrow} \quad Q_1 = (u_2 - u_1\boldsymbol{i}_3) \;\in \mathbb{H}.$$

We see that the complex number imaginary $i \in \mathbb{C}$ must be associated with a unit bivector as the transversal plane unit pseudoscalar $\boldsymbol{i} \rightarrow \boldsymbol{i}_3 \in \mathbb{H}$, that in $\mathfrak{Z}$-space is the unit bivector that spans the chosen transversal plane.

(6.172) $$i \quad \leftrightarrow \quad \boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3 = \boldsymbol{i}_1\boldsymbol{i}_2 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1.$$

In the complex number tradition, the state superposition of the internal spinor orthogonality is expressed as a matrix spinor

(6.173) $$|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}u_0 + i\,u_3\\ u_2 - i\,u_1\end{pmatrix}$$

With this complex number matrix form, we lose its spatial ***direction*** to the transcendental.
The two complex numbers exist in the same mental imaginary plane as an a priori unknown.

The two complex numbers represent two orthogonal oscillating 1-spinors each given by their own angular development parameters

(6.174) $$\alpha = u_0 + i\,u_3 = \varrho e^{+i½\phi} \quad \leftrightarrow \quad Q_3 = (u_0 + u_3\boldsymbol{i}_3) = \varrho e^{+\boldsymbol{i}_3½\phi}$$
$$\beta = u_0 - i\,u_3 = \rho e^{-i½\psi} \quad \leftrightarrow \quad Q_1 = (u_2 - u_1\boldsymbol{i}_3) = \rho e^{-\boldsymbol{i}_3½\psi}$$

The picture that these two complex numbers $\alpha, \beta \in \mathbb{C}$ exist in the same physical geometric plane for an ***entity*** (the complex plane) is *a complete illusion*. Contrary, the geometric quaternion $\hat{Q} = Q_3 1 + Q_1\boldsymbol{i}_2$ picture is more realistic with the circle 1-spinor Q_3 acting on the unit scalar 1 and another circle 1-spinor e.g., Q_1 left acting on the unit bivector ***direction*** $\boldsymbol{i}_2$.
Consult Figure 6.15, Figure 6.16 and Figure 6.11 together with (6.151)-(6.154).[317]
In the matrix tradition, this is expressed in a complex 2×2 matrix

(6.175) $$\begin{bmatrix}\alpha & -\beta^*\\ \beta & \alpha^*\end{bmatrix} = \begin{bmatrix}u_0 + i\,u_3 & -u_2 - i\,u_1\\ u_2 - i\,u_1 & u_0 - i\,u_3\end{bmatrix} \quad \leftrightarrow \quad \hat{Q} = \tfrac{1}{2}\begin{pmatrix}1 & \boldsymbol{i}_2\end{pmatrix}\begin{bmatrix}u_0 + u_3\boldsymbol{i}_3 & -u_2 - u_1\boldsymbol{i}_3\\ u_2 - u_1\boldsymbol{i}_3 & u_0 - u_3\boldsymbol{i}_3\end{bmatrix}\begin{pmatrix}1\\ -\boldsymbol{i}_2\end{pmatrix}.$$

The product of these 2×2 complex matrices stays closed inside the special unitary group $SU(2)$.

6.4.5.2. Two State Observable of a Fundamental *Entity* in $\mathfrak{Z}$ Space

The Stern-Gerlach experiment from 1922 shows two state of Ag atoms in a gradient magnetic field. As a new phenomenon this was interpreted as a two-state wavefunction to the Schrödinger equation I. (2.65) $i\hbar\frac{\partial}{\partial t}|\psi\rangle = \widehat{H}|\psi\rangle$ where $\widehat{H}$ implicit carry the magnetic impact. We count the two stats of the wavefunction as

(6.176) $$|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \qquad \text{spanned from two abstract column basis vectors} \quad |\uparrow\rangle \equiv \begin{pmatrix}1\\ 0\end{pmatrix} \text{ and } |\downarrow\rangle \equiv \begin{pmatrix}0\\ 1\end{pmatrix}.$$

We interpret the idea from Pauli's work as founded in this abstract column vectors basis of the two-state phenomena. The two separate components of this two-dimensional linear vector space over a complex scalar field $\mathbb{C}_1^2 \sim (V_2, \mathbb{C})$ are spanned from this basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ as

(6.177) $$|\psi\rangle = |\psi\!\uparrow\rangle + |\psi\!\downarrow\rangle = \alpha\begin{pmatrix}1\\ 0\end{pmatrix} + \beta\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}\alpha\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}u_0 + i\,u_3\\ u_2 - i\,u_1\end{pmatrix}$$

To find the spatial angular momentum structure of this, the Pauli matrices (6.116) $\hat{\sigma}_3, \hat{\sigma}_2, \hat{\sigma}_1, \hat{\sigma}_0$ had traditionally been used in the literature without intuit capability to describe the spatial ***directions*** of the situated locality of one ***entity***.
Then: Because the *even closed lifted Pauli group* (6.118): $\{\pm\hat{\sigma}_0, \pm i\hat{\sigma}_1, \pm i\hat{\sigma}_2, \pm i\hat{\sigma}_3\}$
is isomorph to the *even closed quaternion group* (6.130): $\{\pm 1, \pm\boldsymbol{i}_1, \pm\boldsymbol{i}_2, \pm\boldsymbol{i}_3\}$, –[318]
we compare this with the formulation with the even geometric algebra in the quaternion form

(6.178) $$U = \hat{Q} = u_0 + u_1\boldsymbol{i}_1 + u_2\boldsymbol{i}_2 + u_3\boldsymbol{i}_3$$

From all this analysis (6.145), (6.162)-(6.163), (6.165)-(6.168), (6.177) we see that an ***entity*** can be described of two independent plane angular circular 1-spinors (6.170)-(6.171).

The strength of the closed multiplication group structure shown in Table 6.2 is
that all linear combined products of elements stay as elements in the group body.
E.g., the quaternions $Q \in \mathbb{H}$ when multiplied and additive combined stays in $\mathbb{H}$.
Special the product of spinors is spinors and the product of bivectors is spinors (two orthogonal bivectors give just a third bivector).
Even the product of two 1-vectors outside the even algebra jumps into the closed lifted algebra as a spinor in $\mathbb{H}$. All the elements of the quaternion group fall into four categories:
Scalars, bivectors, 1-spinors, and 2-spinors.

[317] In Doran & Lasenby' s Geometric algebra for physicist [18]p.271 (8.21) the basis for the eigenstates are named $|\uparrow\rangle \leftrightarrow 1$ and $|\downarrow\rangle \leftrightarrow \boldsymbol{i}\boldsymbol{\sigma}_2 = \boldsymbol{i}_2$. They call them spin-up and spin-down basis stats. My heuristic interpretation is, that the relative orientation between two ***directional*** oscillating 1-spinor stats (6.174) is what gives the impact of two spin ±½ states.

[318] The structure was first instrumentalised by Hamilton as quaternions in 1843- and later reinvented as Pauli matrices 1926-8.

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