the orthogonal ***direction*** to the first by multiplying by the third ***directional*** bivector $\boldsymbol{i}_j$.
To simplify this complex interconnectivity, we recall $\boldsymbol{i}_j = \boldsymbol{i}_k\boldsymbol{i}_l$, (for $l\neq j\neq k\neq l$ ←1,2,3, permutated).
E.g. as (6.126) for which we choose a local reference frame axis $\boldsymbol{\sigma}_3$ with the transversal plane $\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3 = \boldsymbol{i}_1\boldsymbol{i}_2 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$. Hereby we see that (6.145f) is a case. Then from the spinor angles in this first plane $\boldsymbol{i}_3$, that gives the two driving 1-spinors

(6.146) $$Q_3 = (u_0+u_3\boldsymbol{i}_3) = \varrho(\cos\tfrac{1}{2}\phi + \boldsymbol{i}_3\sin\tfrac{1}{2}\phi) = \varrho\, e^{+\boldsymbol{i}_3\frac{1}{2}\phi} = \varrho U_3, \quad \varrho = \sqrt{u_0^2+u_3^2} \le 1, \quad \text{(Figure 6.15)}$$

(6.147) $$Q_1 = (u_2-u_1\boldsymbol{i}_3) = \rho(\cos\tfrac{1}{2}\psi - \boldsymbol{i}_3\sin\tfrac{1}{2}\psi) = \rho\, e^{-\boldsymbol{i}_3\frac{1}{2}\psi} = \rho U_1^\dagger, \quad \rho = \sqrt{u_1^2+u_2^2} \le 1. \quad \text{(Figure 6.16)}$$

where we introduce the two angles in that plane

(6.148) $$\phi = 2\cos^{-1}(u_0/\varrho) = 2\sin^{-1}(u_3/\varrho), \quad \text{hence} \quad u_0 = \varrho\cos\tfrac{1}{2}\phi, \quad u_3 = \varrho\sin\tfrac{1}{2}\phi \quad \text{and}$$

(6.149) $$\psi = 2\cos^{-1}(u_2/\rho) = 2\sin^{-1}(u_1/\rho), \quad \text{hence} \quad u_2 = \rho\cos\tfrac{1}{2}\psi, \quad u_1 = \rho\sin\tfrac{1}{2}\psi, \quad \text{so that}$$

(6.150) $$\varrho^2+\rho^2 = \varrho^2\cos^2\tfrac{1}{2}\phi + \varrho^2\sin^2\tfrac{1}{2}\phi + \rho^2\cos^2\tfrac{1}{2}\psi + \rho^2\sin^2\tfrac{1}{2}\psi = u_0^2+u_3^2+u_1^2+u_2^2 = 1,$$

where the two modulus amplitudes ϱ and ρ merge to unity.
We have here used the ideal circular form as the 1-spinors (6.146) and (6.147)
You could argue to use Kepler-ellipse cause of the four degrees of freedom allow this inside the unitary condition (6.137), but here we don't consider a central force field with any particle, planet, or any distribution of individual particles. Here we are only concerned about the symmetries between plane ***directions*** and their relative ***quantitative*** magnitudes in our foundation of any indivisible ***entity*** Ψ ontological in $\mathfrak{Z}$-space.
This pure angular area view of (6.147) as a circle sector in the transversal plane to the 1-vector $\boldsymbol{\sigma}_3$ for the intuition is insufficient.
We rewrite the last term in (6.145f) and further the same for (6.145e)

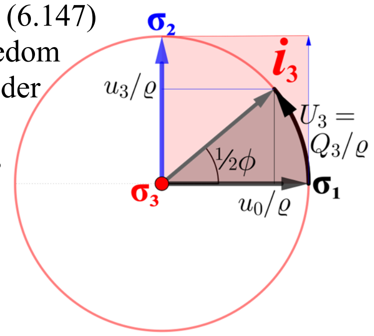

Figure 6.15 1-rotor U_3 for spinor $Q_3 = \varrho U_3$.

(6.151) $$\begin{aligned} Q_1\boldsymbol{i}_2 &= (u_2-u_1\boldsymbol{i}_3)\boldsymbol{i}_2 \\ &= \rho(\cos\tfrac{1}{2}\psi - \boldsymbol{i}_3\sin\tfrac{1}{2}\psi)\boldsymbol{i}_2 = u_2\boldsymbol{i}_2 + u_1\boldsymbol{i}_1 = u_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 - u_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = (u_2\boldsymbol{\sigma}_1 - u_1\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_3 \\ &= \rho\big((\cos\tfrac{1}{2}\psi)\boldsymbol{\sigma}_1 - (\sin\tfrac{1}{2}\psi)\boldsymbol{\sigma}_2\big)\boldsymbol{\sigma}_3 = \rho(\boldsymbol{i}_2\cos\tfrac{1}{2}\psi + \boldsymbol{i}_1\sin\tfrac{1}{2}\psi) \\ &= \rho(\sin\tfrac{1}{2}\psi + \boldsymbol{i}_3\cos\tfrac{1}{2}\psi)\boldsymbol{i}_1 = (u_1+u_2\boldsymbol{i}_3)\boldsymbol{i}_1 = Q_2\boldsymbol{i}_1. \end{aligned}$$

We see that the unit 1-vector term (as a factor in the first part third line)

(6.152) $$\begin{aligned} \mathbf{e}_1 = \hat{\mathbf{r}} &= \big((\cos\tfrac{1}{2}\psi)\boldsymbol{\sigma}_1 - (\sin\tfrac{1}{2}\psi)\boldsymbol{\sigma}_2\big) \\ &= \cos(-\tfrac{1}{2}\psi)\,\boldsymbol{\sigma}_1 + \sin(-\tfrac{1}{2}\psi)\,\boldsymbol{\sigma}_2, \end{aligned}$$

look like the **unit circle** in Cartesian coordinates in the plane of $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\}$ retrograde orientated by $-\tfrac{1}{2}\psi$ (clockwise). But the right operation with $\boldsymbol{\sigma}_3$ turns the subject into the space outside this plane to exist in the plane supported by the unit bivector

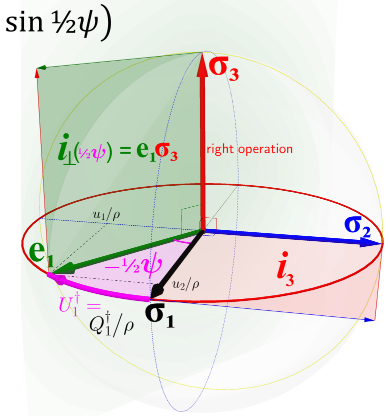

Figure 6.16 The turned plane unit object $\boldsymbol{i}_\perp(\tfrac{1}{2}\psi)$.

(6.153) $$\boldsymbol{i}_\perp(\tfrac{1}{2}\psi) = \mathbf{e}_1\boldsymbol{\sigma}_3 = \boldsymbol{i}_2 U_1 = U_1^\dagger\boldsymbol{i}_2 \qquad = Q_1\boldsymbol{i}_2/\rho = Q_2\boldsymbol{i}_1/\rho,$$

rotating with $\tfrac{1}{2}\psi$ by $U_1^\dagger = e^{-\boldsymbol{i}_3\frac{1}{2}\psi}$ as displayed in Figure 6.16.
The expression (6.151) concerns two of the four quaternion coordinates in (6.145)

(6.154) $$Q_1\boldsymbol{i}_2 = Q_2\boldsymbol{i}_1 = \rho(\boldsymbol{i}_2\cos\tfrac{1}{2}\psi + \boldsymbol{i}_1\sin\tfrac{1}{2}\psi) = +u_1\boldsymbol{i}_1 + u_2\boldsymbol{i}_2 = \rho\boldsymbol{i}_\perp(\tfrac{1}{2}\psi) = \mathrm{B}(\tfrac{1}{2}\psi) = \rho e^{-\boldsymbol{i}_3\frac{1}{2}\psi}\boldsymbol{i}_2.$$

This linear combination of the unit (orthonormal) bivector basis $\{\boldsymbol{i}_1,\boldsymbol{i}_2\}$ results in the turning bivector $\mathrm{B}(\tfrac{1}{2}\psi) = \rho\boldsymbol{i}_\perp(\tfrac{1}{2}\psi) = \rho\mathbf{e}_1\boldsymbol{\sigma}_3$ scaled by ρ in this plane ***direction*** $\boldsymbol{i}_\perp(\tfrac{1}{2}\psi)$, see Figure 6.16.
To intuit the foundation for this unit bivector basis $\{\boldsymbol{i}_1 = \boldsymbol{\sigma}_3\boldsymbol{\sigma}_2, \boldsymbol{i}_2 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_3\}$ compare with Figure 6.11.
The idea of Figure 6.16 is to see the object bivector $\boldsymbol{i}_\perp = \mathbf{e}_1\boldsymbol{\sigma}_3$ as fixed in the external lab frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3 = \boldsymbol{\sigma}_3\}$ and the autonomy frame $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ for a physical ***entity*** Ψ spinning with the oscillator rotor $U_1 = e^{\boldsymbol{i}_3\frac{1}{2}\psi}$: E.g., the three $\boldsymbol{\sigma}_j(\tfrac{1}{2}\psi) = e^{\boldsymbol{i}_3\frac{1}{2}\psi}\mathbf{e}_j$ relative to the laboratory ***directions***.

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Geometric Critique of Pure Mathematical Reasoning
Jens Erfurt Andresen
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Interpreted as oscillators the two 1-spinors Q_1 and Q_2 are interconnected. Reason (6.151),

(6.155) $$Q_1\boldsymbol{i}_2 = Q_2\boldsymbol{i}_1 \quad \Leftrightarrow \quad Q_2 = Q_1\boldsymbol{i}_3, \qquad \text{by right multiplying the first equation with } -\boldsymbol{i}_1.$$

We see that the three oscillators $Q_1\to Q_1\boldsymbol{i}_2$, $Q_2\to Q_2\boldsymbol{i}_1$ and Q_3 are orthogonal, and that one Q_3 is an angular phase independent of the two others. These two then are mutually interconnected by the chosen ***direction*** $\boldsymbol{i}_3$ (6.155) with the phase angle dependency as an example for our intuition. We will look further into the idea of this fundamental issue on the next pages 259-308.

We will now interpret the identity $\boldsymbol{i}_1 = \boldsymbol{i}_2\boldsymbol{i}_3 = \boldsymbol{i}_3(-\boldsymbol{i}_2)$ as a rotation of the plane bivector $\boldsymbol{i}_3$ into the plane bivector $\boldsymbol{i}_1$ by the left operator $\boldsymbol{i}_2$ or the right operation by $-\boldsymbol{i}_2$.
In the same way, we interpret the identity $\boldsymbol{i}_2 = \boldsymbol{i}_3\boldsymbol{i}_1 = -\boldsymbol{i}_1\boldsymbol{i}_3$ as a rotation of the plane bivector $\boldsymbol{i}_3$ into the plane bivector $\boldsymbol{i}_2$ by the left operator $-\boldsymbol{i}_1$ or the right operation by $\boldsymbol{i}_1$.
Here we interpret these operators as a perpendicular rotation by rotors in the same planes

(6.156) $$\pm\boldsymbol{i}_2 = e^{\pm\boldsymbol{i}_2\pi/2} = \boldsymbol{i}_2\sin(\pm\pi/2), \qquad \text{in the plane, spanned from } \{\boldsymbol{i}_2\}\sim\{\boldsymbol{\sigma}_3,\boldsymbol{\sigma}_1\}.$$

(6.157) $$\pm\boldsymbol{i}_1 = e^{\pm\boldsymbol{i}_1\pi/2} = \boldsymbol{i}_1\sin(\pm\pi/2), \qquad \text{in the plane, spanned from } \{\boldsymbol{i}_1\}\sim\{\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}.$$

To make a perpendicular regular rotation along[315] these planes we invent the rotors

(6.158) $$U_{\boldsymbol{i}_2} = e^{\boldsymbol{i}_2\frac{1}{2}\pi/2} = e^{\boldsymbol{i}_2\pi/4} = \cos(\pi/4) + \boldsymbol{i}_2\sin(\pi/4) = \sqrt{\tfrac{1}{2}}(1+\boldsymbol{i}_2).$$

(6.159) $$U_{-\boldsymbol{i}_1} = e^{-\boldsymbol{i}_1\pi/4} = \cos(-\pi/4) + \boldsymbol{i}_1\sin(-\pi/4) = \sqrt{\tfrac{1}{2}}(1-\boldsymbol{i}_1).$$

With the canonical rotations by these, we tilt the plane bivector $\boldsymbol{i}_3$ ***direction*** into the $\boldsymbol{i}_1$ ***direction***

(6.160) $$\boldsymbol{i}_3 \longrightarrow U_{\boldsymbol{i}_2}\boldsymbol{i}_3U_{\boldsymbol{i}_2}^\dagger = \sqrt{\tfrac{1}{2}}(1+\boldsymbol{i}_2)\,\boldsymbol{i}_3(1-\boldsymbol{i}_2)\sqrt{\tfrac{1}{2}} = \boldsymbol{i}_1 \quad = \boldsymbol{i}_2\boldsymbol{i}_3 = \boldsymbol{i}_3(-\boldsymbol{i}_2), \qquad U_{\boldsymbol{i}_2}^2 = \boldsymbol{i}_2.$$

and tilt the plane bivector ***direction*** $\boldsymbol{i}_3$ into the plane bivector $\boldsymbol{i}_2$ ***direction*** by a $U(1)$ 1-rotor

(6.161) $$\boldsymbol{i}_3 \longrightarrow U_{-\boldsymbol{i}_1}\boldsymbol{i}_3U_{-\boldsymbol{i}_1}^\dagger = \sqrt{\tfrac{1}{2}}(1-\boldsymbol{i}_1)\,\boldsymbol{i}_3(1+\boldsymbol{i}_1)\sqrt{\tfrac{1}{2}} = \boldsymbol{i}_2 \quad = -\boldsymbol{i}_1\boldsymbol{i}_3 = \boldsymbol{i}_3\boldsymbol{i}_1, \qquad U_{-\boldsymbol{i}_1}^2 = -\boldsymbol{i}_1.$$

By this mapping perpendicular to the reference 1-spinor Q_3 in the $\boldsymbol{i}_3$ plane, we from Q_1 (6.147) and (6.145f) define its orthogonal 1-spinor[316] turned by $\boldsymbol{i}_2 = e^{\boldsymbol{i}_2\pi/2}$

(6.162) $$Q_1^\perp = U_{\boldsymbol{i}_2}Q_1U_{\boldsymbol{i}_2}^\dagger = (u_2-u_1\boldsymbol{i}_1) = \quad \rho(\cos\tfrac{1}{2}\psi - \boldsymbol{i}_1\sin\tfrac{1}{2}\psi) = \rho e^{-\boldsymbol{i}_1\frac{1}{2}\psi}.$$

And from Q_2 we get the other perpendicular 1-spinor from (6.145e) turned $-\boldsymbol{i}_1 = e^{-\boldsymbol{i}_1\pi/2}$

(6.163) $$\begin{aligned} Q_2^\perp = U_{-\boldsymbol{i}_1}Q_2U_{-\boldsymbol{i}_1}^\dagger = (u_1+u_2\boldsymbol{i}_2) &= \rho(\sin\tfrac{1}{2}\psi + \boldsymbol{i}_2\cos\tfrac{1}{2}\psi) = \rho(\cos\tfrac{1}{2}\phi_2 + \boldsymbol{i}_2\sin\tfrac{1}{2}\phi_2) \\ \text{(by the reason)} \qquad &= -\boldsymbol{i}_2\rho(\cos\tfrac{1}{2}\psi + \boldsymbol{i}_2\sin\tfrac{1}{2}\psi) = \rho e^{-\boldsymbol{i}_2\pi/2}e^{\boldsymbol{i}_2\frac{1}{2}\psi} = \rho e^{\boldsymbol{i}_2\frac{1}{2}\phi_2}, \end{aligned}$$

where we introduce $\phi_2 = \psi - \pi$ with a rotation phase angle shift $-\tfrac{1}{2}\pi$ as a factor $-\boldsymbol{i}_2 = e^{-\boldsymbol{i}_2\pi/2}$ in the same plane as the 1-spinor $Q_2^\perp$ (where they commute). (By ⊥ we mean perpendicular to Q_3)

These two 1-spinors (6.162) and (6.163) are perpendicular to each other $Q_1^\perp \perp Q_2^\perp$ just as the transversal bivector planes $\boldsymbol{i}_1 \perp \boldsymbol{i}_2$ are orthogonal $\boldsymbol{i}_1\cdot\boldsymbol{i}_2 = 0$.
It is obvious that these two simple plane 1-spinors $Q_1^\perp$ and $Q_2^\perp$ represent circle oscillators by the synchronous development parameters $\phi_2+\pi = \psi \sim \omega t$ in an interconnected relationship to the plane ***direction*** $\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3 = \boldsymbol{i}_1\boldsymbol{i}_2 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ transversal to the external lab ***direction*** $\boldsymbol{\sigma}_3 \coloneqq \|\mathbf{e}_3$

– A extrapolation to some ***direction*** components of oscillating 'wavefunctions':
When we later will consider these 1-spinors as synchronous circle oscillators, we can imagine some phase shift between three oscillating 1-spinors Q_3, $Q_2^\perp$, $Q_1^\perp$ joined with the three quaternion basis plane ***directions*** $\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3$. A picture of this is, that we have three orthogonal 1-spinors:

[315] We have to consider the idea of the plane ***directions*** $\boldsymbol{i}_2 \equiv \boldsymbol{\sigma}_1\boldsymbol{\sigma}_3$ and $\boldsymbol{i}_1 \equiv \boldsymbol{\sigma}_3\boldsymbol{\sigma}_2$ as translation invariant subjects over the entire $\mathfrak{Z}$ space of some intuited plane objects $\{\boldsymbol{\sigma}_3,\boldsymbol{\sigma}_1\}$ and $\{\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ that we use for our distinction of the ***directions***.
[316] We here remember that these (trigonometric) scalars u_1, u_2 are invariant under rotations. The ***direction*** $\boldsymbol{i}_3 \to \boldsymbol{i}_1$ chance.

For quotation reference use: ISBN-13: 978-8797246931