Therefore, the magnitude of the quaternion is $\alpha = |Q| = \sqrt{q_0^2+q_1^2+q_2^2+q_3^2} \ \in\mathbb{R}$.
The symmetric bilinear form for the two generating bivectors (6.126) gives $(\boldsymbol{i}_1)^2 = -1$ and $(\boldsymbol{i}_2)^2 = -1$, therefore a geometric algebra named $\mathcal{G}_{0,2}$ just as a Clifford algebra $C\ell_{0,2}$.[312]
The third bivector ***direction*** is then implicit given $\boldsymbol{i}_3 := \boldsymbol{i}_1\boldsymbol{i}_2$, where $\boldsymbol{i}_3^2 = -1$, and the scalar ***quality*** is introduced by the product $1 = \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1$, hence a full closed even algebra $\mathcal{G}_{0,2}$.
In our practice, the negative part[87] of the real field of quaternions coordinates $q_k < 0$ make the negative part of the basis (6.130) unnecessary. The necessary standard basis for $\mathcal{G}_{0,2}(\mathbb{R})$ is just

(6.134) $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\} \quad = \quad \{+1 = \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1,\ \ \boldsymbol{i}_1 = \boldsymbol{i}_2\boldsymbol{i}_3,\ \ \boldsymbol{i}_2 = \boldsymbol{i}_3\boldsymbol{i}_1,\ \ \boldsymbol{i}_3 = \boldsymbol{i}_1\boldsymbol{i}_2\}.$

The linear algebraic space $\mathcal{G}_{0,2}$ is spanned by a real field $q_\mu \in\mathbb{R}$, μ=0,1,2,3, so we could say that $\mathcal{G}_{0,2}(\mathbb{R})$ has four dimensions of real linearity, and note the even mixed ***grades*** of dimensionalities.
We see the three ***directions*** $\boldsymbol{i}_k$ of this basis set have the ***primary quality of second grade*** (***pqg***-2).
We note that the one-dimensional scalar q_0 ***of zero grade*** (***pqg***-0) is ***non-directional***.
From this (6.131), (6.134) we deduce that the conceptual quaternion $\mathbb{H}\sim\mathcal{G}_{0,2}(\mathbb{R})$ idea describes the full three dimension ***directional*** relations of ***primary quality of second grade*** (***pqg***-2) in $\mathfrak{Z}$-space by the even closed geometric algebra $\mathcal{G}_{0,2}(\mathbb{R})$, which we call the *quaternion algebra*.
When it comes to the ***direction*** $\hat{Q} = Q/|Q|$ of a quaternion $Q \in\mathbb{H}$ there is only one real dilation factor for its magnitude α=$|Q|$. This supporting ***pqg***-2-***direction*** has only one linear real dimension

(6.135) $Q = \alpha\hat{Q} \in\mathbb{H}, \quad \alpha \in\mathbb{R}.$

6.4.4. The Versor as a *Direction* 2-Rotor

A unitary **quaternion** was by Hamilton called a *versor*, we in this book also call it a *2-rotor*

(6.136) $U = \hat{Q} = Q/|Q| = u_0+u_1\boldsymbol{i}_1+u_2\boldsymbol{i}_2+u_3\boldsymbol{i}_3 \in \mathbb{H}, \quad$ where $\forall u_\kappa \in\mathbb{R}$ for $\kappa = 0,1,2,3$. (norm one)

For such a real field unit quaternion 2-rotor group, we demand unit spherical S^3 symmetry[313]

(6.137) $UU^\dagger = 1 = u_0^2+u_1^2+u_2^2+u_3^2,$

6.4.4.2. The Traditional View on the Spatial Rotation Problem

To make the following intuition we make a *false presume* that U is never a pure real scalar, i.e., $u_0^2 < 1$ so that not all ***directional*** real components disappear $u_1^2+u_2^2+u_3^2 > 0$. Opposite when this happens $u_1^2 = u_2^2 = u_3^2 = 0 \Rightarrow u_0^2 = 1$ the ***direction*** part disappear, but essential $u_\kappa u_\kappa = 1$ for U.
When we prevent the ***direction*** free case, we can separate the bivector part (6.125) as

(6.138) $\mathrm{B} = \boldsymbol{\imath}\mathrm{b} = u_1\boldsymbol{i}_1+u_2\boldsymbol{i}_2+u_3\boldsymbol{i}_3$, $\qquad$ from the versor (6.136) $U = u_0 + \mathrm{B}$.

transversal to its dual 1-vector b. We normalize this to a unit ***pqg-2-direction*** bivector

(6.139) $\boldsymbol{\imath}\mathrm{n} := (u_1\boldsymbol{i}_1+u_2\boldsymbol{i}_2+u_3\boldsymbol{i}_3)/|u|$, where $u^2 = 1-u_0^2 = u_1^2+u_2^2+u_3^2$, and $0 < |u| \le 1$, $u \in\mathbb{R}$,

as transversal to its dual 1-vector ***direction*** (mediated through the unit chiral volume $\boldsymbol{\imath}$)

(6.140) $\mathrm{n} := (u_1\sigma_1+u_2\sigma_2+u_3\sigma_3)/\sqrt{1-u_0^2} = n_1\sigma_1+n_2\sigma_2+n_3\sigma_3 \quad \in(V_3,\mathbb{R}),$ where $|\mathrm{n}|=1$.[314]

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[312] $\mathcal{G}_{0,2}\sim\mathcal{G}(V_2,\mathbb{R})$, here $\dim V_2 = 2$ indicate two bivector directions, e.g. $\boldsymbol{i}_1, \boldsymbol{i}_2$ as supporting generators (6.126) with $q_1\boldsymbol{i}_1 + q_2\boldsymbol{i}_2$, while the third direction is implicit interconnected $\boldsymbol{i}_3 \equiv \boldsymbol{i}_1\boldsymbol{i}_2$, $\boldsymbol{i}_3^2 = -1$ for the 1-spinor $q_0 1 + q_3\boldsymbol{i}_3$, where the scalar unit $1 = \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1$ is introduced, (these $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ can be cyclically permutated). The unitary $U(1)$ circle group of elements $e^{i\varphi} \in\mathbb{T}\subset\mathbb{C}$ is a one-parameter group, real argued by $\varphi \in\mathbb{R}$ as an angular measure. We will below see that such two angular parameters can determine a quaternion direction in a S^2 unit sphere (spherical coordinates), therefore we have $C\ell_{0,2}(V_2,\mathbb{C})\sim\mathcal{G}_{0,2}(\mathbb{R})$, isomorph with $SU(2)$. The two bivectors $\boldsymbol{i}_k$ and $\boldsymbol{i}_j$ where $k \ne j$ will be generators for the angular development in two rotors $e^{\boldsymbol{i}_k\varphi_k}$ and $e^{\boldsymbol{i}_j\varphi_j}$. Two other reals can do the dilation of these unitary rotors, in all four reals. The heuristic picture in $\mathbb{C}^2$ seems ambiguous.
[313] These simple real quaternions are Hermitian. For a rotor, we use $U^\dagger = \widetilde{U}$ to match the form from quantum mechanics.
[314] All these 1-vectors $\forall\mathrm{n} \in(V_3,\mathbb{R})$ has all ***direction*** possibilities in the $\mathfrak{Z}$ space spherical S^2-symmetry $n_1^2+n_2^2+n_3^2=1$. The new is that the versor quaternion is a mixed linear dimensionality of four reals: $\dim(\forall\hat{Q} \in\mathbb{H}\sim\mathrm{span}((6.126)))\sim 4$, with a S^3-symmetry $u_0^2+u_1^2+u_2^2+u_3^2=1$ for the special unitary group $SU(2)$ which homomorphy to the orthogonal rotation group $O^+(3)\sim SO(3)$.

We have here introduced the Cartesian coordinates for the unit 1-vector ***direction***

(6.141) $\left(n_1 = \frac{u_1}{\sqrt{1-u_0^2}},\ \ n_2 = \frac{u_2}{\sqrt{1-u_0^2}},\ \ n_3 = \frac{u_3}{\sqrt{1-u_0^2}}\right).$

Thus, we write the specific ***direction*** unit bivector

(6.142) $\boldsymbol{i}_{\mathrm{n}} = \boldsymbol{\imath}\mathrm{n} = n_1\boldsymbol{i}_1+n_2\boldsymbol{i}_2+n_3\boldsymbol{i}_3, \qquad$ where $\boldsymbol{i}_{\mathrm{n}}^2 = -1 \Rightarrow |\boldsymbol{i}_{\mathrm{n}}| = 1$ and $n_1^2+n_2^2+n_3^2 = 1.$

Then we have the 1-rotor form

(6.143) $U = u_0+u\boldsymbol{i}_{\mathrm{n}},$

with the bivector dilation $|u| = \sqrt{u_1^2+u_2^2+u_3^2}$. We choose the scalar part $u_0 = \cos\frac{1}{2}\varphi$, so that $\varphi = 2\cos^{-1}(u_0)$, and hence we have for the transversal bivector part $u = \sin\frac{1}{2}\varphi$.
Whence the rotor for the ***direction*** (6.139) and (6.142) is

(6.144) $U = u_0 + u\boldsymbol{\imath}\mathrm{n} = \cos\frac{1}{2}\varphi + \boldsymbol{\imath}\mathrm{n}\sin\frac{1}{2}\varphi = U_{\boldsymbol{\imath}\mathrm{n}} = e^{\boldsymbol{\imath}\mathrm{n}\frac{1}{2}\varphi} = e^{(n_1\boldsymbol{i}_1+n_2\boldsymbol{i}_2+n_3\boldsymbol{i}_3)\frac{1}{2}\varphi} \ \in\mathbb{H}.$

We see that there is a problem with the 1-rotor form $u_0+u\boldsymbol{i}_{\mathrm{n}}$ because the transversal bivector ***direction*** $\boldsymbol{i}_{\mathrm{n}} = \boldsymbol{\imath}\mathrm{n}$ is governed by three arguments (6.139)-(6.142) from the fundamental form (6.136). This gives extra freedom of ***direction*** fluctuation inside the $\mathcal{G}_{0,2}$ algebra.
The ***direction*** $\boldsymbol{i}_{\mathrm{n}}$ unit condition (6.142) demands ***direction*** continuity at $u_0 \to 1$ and $u \to 0$.
Anyway, for the intuition, the simple even graded 1-rotor form idea (6.143) $U = u_0+u\boldsymbol{i}_{\mathrm{n}}$ is necessary for the idea of a bivector exponential development $e^{\boldsymbol{i}_{\mathrm{n}}\frac{1}{2}\varphi}$ in its own transversal ***direction*** $\boldsymbol{i}_{\mathrm{n}} = \boldsymbol{\imath}\mathrm{n}$ as the ***primary quality of second grade*** in duality with the odd ***first grade direction*** $\mathrm{n} = -\boldsymbol{\imath}\boldsymbol{i}_{\mathrm{n}}$, by intervention of the ***third grade*** chiral volume pseudoscalar $\boldsymbol{\imath}$.
The extra freedom with fluctuation of this intuit rotation axis ***direction*** we shall investigate further.
This could be a second or several independent exponential development 1-rotors.
We keep track of the essential even algebra $\mathcal{G}_{0,2}$ and look at a 2-rotor idea of the versor quaternion ***quality*** and keep in mind the duality to understand all the aspects of the ***grades pqg***-0, ***pqg***-1, ***pqg***-2 and ***pqg***-3 in combination, and ask the question: What does this mean in the perspective of the mixed basis (6.119) with the later formulation (6.179)? First, we can say, that the coordinates u_1, u_2, u_3 (6.136) or the non-zero (n_1, n_2, n_3) (6.141) determine the ***pqg***-1 ***direction*** $\mathrm{n} = (u_1\sigma_1+u_2\sigma_2+u_3\sigma_3)/|u|$, and further that the rotation angle is $\varphi = 2\cos^{-1}(u_0)$ in the ***pqg***-2 plane $\boldsymbol{i}_{\mathrm{n}} = \boldsymbol{\imath}\mathrm{n}$. – But in all, for the versor $U = \hat{Q}$ it depends on all four u_0, u_1, u_2, u_3.

6.4.4.3. The Four Real Scalar Coordinates for the Versor Quaternion *Direction*

We take an external view to this (6.144) from (6.136) we have a unit **quaternion** versor $\hat{Q}$ as this ***directed*** 2-*rotor* in an even algebra

(6.145) $U = \hat{Q} = u_0 + u_1\boldsymbol{i}_1 + u_2\boldsymbol{i}_2 + u_3\boldsymbol{i}_3 = u_0 + u\boldsymbol{\imath}\mathrm{n} = U_{\boldsymbol{\imath}\mathrm{n}} = e^{\boldsymbol{\imath}\mathrm{n}\frac{1}{2}\varphi} \in \mathbb{H}\sim\mathcal{G}_{0,2}(\mathbb{R}).$

From this idea, we can separate that versor quaternion in two orthogonal rotation plane ***directions*** along $\boldsymbol{i}_j$ and $\boldsymbol{i}_k$, where $j, k = 1,2,3$ and $k \ne j$, by using the following simple linear algebra technic to separate the independency from the interconnectivity. We have six cases:

(6.145a) $\hat{Q} = u_0+u_1\boldsymbol{i}_1+u_2\boldsymbol{i}_2+u_3\boldsymbol{i}_3 = (u_0+u_1\boldsymbol{i}_1)+(u_2\boldsymbol{i}_2+u_3\boldsymbol{i}_1\boldsymbol{i}_2) = (u_0+u_1\boldsymbol{i}_1)1 + (u_2+u_3\boldsymbol{i}_1)\boldsymbol{i}_2,$
(6.145b) $\hat{Q} = u_0+u_1\boldsymbol{i}_1+u_3\boldsymbol{i}_3+u_2\boldsymbol{i}_2 = (u_0+u_1\boldsymbol{i}_1)+(u_3\boldsymbol{i}_3-u_2\boldsymbol{i}_1\boldsymbol{i}_3) = (u_0+u_1\boldsymbol{i}_1)1 + (u_3-u_2\boldsymbol{i}_1)\boldsymbol{i}_3,$
(6.145c) $\hat{Q} = u_0+u_2\boldsymbol{i}_2+u_3\boldsymbol{i}_3+u_1\boldsymbol{i}_1 = (u_0+u_2\boldsymbol{i}_2)+(u_3\boldsymbol{i}_3+u_1\boldsymbol{i}_2\boldsymbol{i}_3) = (u_0+u_2\boldsymbol{i}_2)1 + (u_3+u_1\boldsymbol{i}_2)\boldsymbol{i}_3,$
(6.145d) $\hat{Q} = u_0+u_2\boldsymbol{i}_2+u_1\boldsymbol{i}_1+u_3\boldsymbol{i}_3 = (u_0+u_2\boldsymbol{i}_2)+(u_1\boldsymbol{i}_1-u_3\boldsymbol{i}_2\boldsymbol{i}_1) = (u_0+u_2\boldsymbol{i}_2)1 + (u_1-u_3\boldsymbol{i}_2)\boldsymbol{i}_1,$
(6.145e) $\hat{Q} = u_0+u_3\boldsymbol{i}_3+u_1\boldsymbol{i}_1+u_2\boldsymbol{i}_2 = (u_0+u_3\boldsymbol{i}_3)+(u_1\boldsymbol{i}_1+u_2\boldsymbol{i}_3\boldsymbol{i}_1) = (u_0+u_3\boldsymbol{i}_3)1 + (u_1+u_2\boldsymbol{i}_3)\boldsymbol{i}_1 = Q_3 1+Q_2\boldsymbol{i}_1,$
(6.145f) $\hat{Q} = u_0+u_3\boldsymbol{i}_3+u_2\boldsymbol{i}_2+u_1\boldsymbol{i}_1 = (u_0+u_3\boldsymbol{i}_3)+(u_2\boldsymbol{i}_2-u_1\boldsymbol{i}_3\boldsymbol{i}_2) = (u_0+u_3\boldsymbol{i}_3)1 + (u_2-u_1\boldsymbol{i}_3)\boldsymbol{i}_2 = Q_3 1+Q_1\boldsymbol{i}_2.$

We presume there are two driving 1-spinor quaternions in the same rotation plane $\boldsymbol{i}_j$, that are complexified in the quaternion by relative creating the orthogonal rotation plane $\boldsymbol{i}_k$. (intuit $\boldsymbol{i}_k \perp \boldsymbol{i}_j$)
– This implicates us to intuit, two independent (orthogonal) perpendicular rotation planes:
The one ***direction*** with the two driving 1-spinors, the first that stays in that ***direction*** by multiplying by 1 and the other perpendicular 1-spinor generated by the second turned in