(6.120) $$A = \underbrace{\alpha}_{pqg\text{-}0} + \underbrace{x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3}_{3\text{D } pqg\text{-}1} + \underbrace{\beta_1\boldsymbol{\imath}\boldsymbol{\sigma}_1 + \beta_2\boldsymbol{\imath}\boldsymbol{\sigma}_2 + \beta_3\boldsymbol{\imath}\boldsymbol{\sigma}_3}_{3\text{D } pqg\text{-}2} + \underbrace{\upsilon\boldsymbol{\imath}}_{pqg\text{-}3,} \quad = \alpha + \mathbf{x} + \mathbf{b}\boldsymbol{\imath} + \upsilon\boldsymbol{\imath} \;. \qquad A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \langle A\rangle_3$$

where $\alpha, x_k, \beta_k, \upsilon \in \mathbb{R}$. We see by comparing (6.119) with (6.117) why we call the local orthonormal basis for the geometric algebra $\mathcal{G}_{3,0}$ with sigma names $\boldsymbol{\sigma}_1 \leftrightarrow \hat{\sigma}_1$, $\boldsymbol{\sigma}_2 \leftrightarrow \hat{\sigma}_2$, $\boldsymbol{\sigma}_3 \leftrightarrow \hat{\sigma}_3$, just as the Pauli matrices as generators for the algebra of the Pauli group.
There is a formal difference in the orientation of these two algebraic structures $\boldsymbol{\imath} \coloneqq \boldsymbol{\sigma}_3\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \leftrightarrow -i$, due to the left sequential multiplication operational definition of $\boldsymbol{\imath}$ in the context of this book.
We can find the local Pauli frame from a dextral reference basis $\{\mathbf{e}_j\}$ by rotation as (6.92), (6.96)

(6.121) $$\boldsymbol{\sigma}_j = U\mathbf{e}_j U^\dagger.$$

The general ***pqg***-1-vectors is like (6.29) given by local coordinates $\mathbf{x} = x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3$
In $\mathcal{G}_{3,0}$ we have the positive norm with $\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_3^2 = 1$ where the quadratic form (6.99)

(6.122) $$\mathbf{x}^2 = (x_1\boldsymbol{\sigma}_1)^2 + (x_2\boldsymbol{\sigma}_2)^2 + (x_3\boldsymbol{\sigma}_3)^2 = x_1^2 + x_2^2 + x_3^2,$$

gives a quadratic measure for the distance, length, or magnitude $d = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

6.4.3. The Quaternion Picture

6.4.3.1. An Anti-Euclidean Geometric Algebra $\mathcal{G}_{0,2}$

We now look at the dual basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\} = \{\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2,\ \boldsymbol{\sigma}_1\boldsymbol{\sigma}_3,\ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\}$ defined from (6.31) in $\mathcal{G}_3(\mathbb{R})$.
From these, we have further the fundamental multivector product *interconnectivity*

(6.123) $$\boxed{\begin{aligned} \boldsymbol{i}_1 &\coloneqq \boldsymbol{\sigma}_3\boldsymbol{\sigma}_2 = -\boldsymbol{i}_3\boldsymbol{i}_2 = \boldsymbol{i}_2\boldsymbol{i}_3 \\ \boldsymbol{i}_2 &\coloneqq \boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 = -\boldsymbol{i}_1\boldsymbol{i}_3 = \boldsymbol{i}_3\boldsymbol{i}_1 \\ \boldsymbol{i}_3 &\coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = -\boldsymbol{i}_2\boldsymbol{i}_1 = \boldsymbol{i}_1\boldsymbol{i}_2 \end{aligned}} \qquad \text{for the unit basis elements in } \mathcal{G}_{0,2}(\mathbb{R}) \subset \mathcal{G}_3(\mathbb{R}).$$

These orthonormal *bivector basis* elements have the ***quality***, that an auto multiplication operation as a quadratic form of the orthogonal plane ***directions*** with negative signature $(-)$ in $\mathcal{G}_{0,2}(\mathbb{R})$

(6.124) $$\boldsymbol{i}_1^2 = \boldsymbol{i}_2^2 = \boldsymbol{i}_3^2 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -1$$

A question, what inversible ***direction*** represent the triple product $\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -1$? Written out in 1-vector basis $\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 \coloneqq \boldsymbol{\sigma}_3\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \boldsymbol{\imath}\boldsymbol{\imath} = \boldsymbol{\imath}^2 = -1$, as the square of the *unit chiral volume* $\boldsymbol{\imath}$ having two orientations of its ***direction*** for the 3-space of physics. For intuition see Figure 6.11&8.
Every bivector plane subject in 3-space can be spanned from a basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ as (6.62)

(6.125) $$\mathrm{B} = \beta_1\boldsymbol{i}_1 + \beta_2\boldsymbol{i}_2 + \beta_3\boldsymbol{i}_3 = \beta_1\boldsymbol{\imath}\boldsymbol{\sigma}_1 + \beta_2\boldsymbol{\imath}\boldsymbol{\sigma}_2 + \beta_3\boldsymbol{\imath}\boldsymbol{\sigma}_3 = \mathbf{b}\boldsymbol{\imath} = \langle \mathrm{B}\rangle_2 \quad \in \mathcal{G}_{0,2}(\mathbb{R}) \quad \subset \mathcal{G}_3(\mathbb{R}).$$

Such even ***pqg***-2 elements $\langle A\rangle_2$ can represent a rotation ***direction*** by the unit $\boldsymbol{i}_{\mathrm{B}} = \hat{\mathrm{B}} = \mathrm{B}/|\mathrm{B}|$.
The strong *interconnectivity* in (6.123) makes us rationalise and choose one ***direction*** of $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ or $\boldsymbol{\sigma}_3$ as a view for intuition. E.g., $\boldsymbol{\sigma}_3 = \hat{\boldsymbol{\omega}}$, where we see the transversal plane $\boldsymbol{i}_3 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \boldsymbol{\imath}\hat{\boldsymbol{\omega}}$ of this 1-vector ***direction***. For the orthogonal space to this transversal plane, we have two independent orthogonal plane basis bivectors $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$, that support this space through the 3-space of physics. We look through the picture of this transversal plane $\boldsymbol{i}_3 = \boldsymbol{i}_1\boldsymbol{i}_2$ of 3-space and have defined a mixed basis

(6.126) $$\{1,\ \boldsymbol{i}_1, \boldsymbol{i}_2,\ \boldsymbol{i}_3 \coloneqq \boldsymbol{i}_1\boldsymbol{i}_2\},$$

that due to (6.124) form an anti-Euclidean geometric algebra $\mathcal{G}_{0,2}(\mathbb{R})$ in 3-space of physics.[309]
This is generated from the bivector basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2\}$, where the $\boldsymbol{i}_3$ ***direction*** is implicit given as orthogonal transversal to the ***pqg***-1 intersection ***direction*** between the two plane ***directions*** $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$.
The idea of this concept is to manage rotation around $\boldsymbol{\sigma}_3$, in the $\boldsymbol{i}_3$ plane.[310]

[309] There are two independent plane ***directions*** $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$ in (6.126), therefore $_{0,2}$ in the designation $\mathcal{G}_{0,2}(\mathbb{R})$ for this algebra.
[310] To get an intuition of this ***pqg***-2 dual space to the ***pqg***-1 ***direction*** the reader can look at Figure 6.11, Figure 6.3 and Figure 6.1,u ($-\boldsymbol{\imath}\boldsymbol{i}_1 = \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 = \mathrm{n}_1$, $-\boldsymbol{\imath}\boldsymbol{i}_2 = \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2 = \mathrm{n}_2$, $-\boldsymbol{\imath}\boldsymbol{i}_3 = \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_3 = \mathrm{n}_3$). To intuit this rotation the reader can take a view at Figure 6.12 and compare to the inclination of the two planes ($\boldsymbol{\imath}\mathrm{n}_1, \boldsymbol{\imath}\mathrm{n}_2$) in Figure 6.1,t.

In duality to the Euclidean space spanned from a standard 1-vector basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ for $\mathcal{G}_{3,0}(\mathbb{R})$ this generalised anti-Euclidean even geometric algebra $\mathcal{G}_{0,2}(\mathbb{R})$ for space spanned by the planes of the supporting bivector basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$. For this we have the two opposite chiral orientations

(6.127) $$\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = \boldsymbol{i}_3\boldsymbol{i}_1\boldsymbol{i}_2 = \boldsymbol{i}_2\boldsymbol{i}_3\boldsymbol{i}_1 = -1 \qquad \text{for the sinistral, inverse to the reversed sequence}$$

(6.128) $$\boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1 = \boldsymbol{i}_1\boldsymbol{i}_3\boldsymbol{i}_2 = \boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{i}_3 = +1 \qquad \text{for the dextral orientation.} \qquad \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1 = \widetilde{\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3}.$$

The foundation of the duality is in the defined pseudoscalar (6.22) $\boldsymbol{\imath} \coloneqq \boldsymbol{\sigma}_3\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ of the Pauli basis (6.119). Here the reader should note that the ***quality*** of the unit chiral volume pseudoscalar possesses a commutative ***quantity*** in the idea of the formulation

(6.129) $$\boldsymbol{\imath} = \sqrt{\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3} = \sqrt{-1}, \quad \text{in that} \quad \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = \boldsymbol{\imath}\boldsymbol{\imath} = -1,$$

The anti-commuting three plane bivector basis elements $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ of the anti-Euclidean even real algebra $\mathcal{G}_{0,2}(\mathbb{R})$ form a multiplicative group with, in all $2^3 = 8$ group elements

(6.130) $$\{+1 = \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1,\ \ \boldsymbol{i}_1 = \boldsymbol{i}_2\boldsymbol{i}_3,\ \ \boldsymbol{i}_2 = \boldsymbol{i}_3\boldsymbol{i}_1,\ \ \boldsymbol{i}_3 = \boldsymbol{i}_1\boldsymbol{i}_2,\ \ -\boldsymbol{i}_1 = \boldsymbol{i}_3\boldsymbol{i}_2,\ \ -\boldsymbol{i}_2 = \boldsymbol{i}_1\boldsymbol{i}_3,\ \ -\boldsymbol{i}_3 = \boldsymbol{i}_2\boldsymbol{i}_1,\ \ -1 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3\}.$$

This group is often called the *Quaternion Group*, which is isomorph with the *Lifted Pauli Group*.
We can write a table of multiplication structures for the possible products inside this closed group:

Table 6.2 Multiplication table for the basis elements of the Quaternion Group

left * right	1	$\boldsymbol{i}_1$	$\boldsymbol{i}_2$	$\boldsymbol{i}_3$	-1	$-\boldsymbol{i}_1$	$-\boldsymbol{i}_2$	$-\boldsymbol{i}_3$
1	1	$\boldsymbol{i}_1$	$\boldsymbol{i}_2$	$\boldsymbol{i}_3$	-1	$-\boldsymbol{i}_1$	$-\boldsymbol{i}_2$	$-\boldsymbol{i}_3$
$\boldsymbol{i}_1$	$\boldsymbol{i}_1$	-1	$\boldsymbol{i}_3$	$-\boldsymbol{i}_2$	$-\boldsymbol{i}_1$	1	$-\boldsymbol{i}_3$	$\boldsymbol{i}_2$
$\boldsymbol{i}_2$	$\boldsymbol{i}_2$	$-\boldsymbol{i}_3$	-1	$\boldsymbol{i}_1$	$-\boldsymbol{i}_2$	$\boldsymbol{i}_3$	1	$-\boldsymbol{i}_1$
$\boldsymbol{i}_3$	$\boldsymbol{i}_3$	$\boldsymbol{i}_2$	$-\boldsymbol{i}_3$	-1	$-\boldsymbol{i}_3$	$-\boldsymbol{i}_2$	$\boldsymbol{i}_1$	1
-1	-1	$-\boldsymbol{i}_1$	$-\boldsymbol{i}_2$	$-\boldsymbol{i}_3$	1	$\boldsymbol{i}_3$	$\boldsymbol{i}_2$	$\boldsymbol{i}_3$
$-\boldsymbol{i}_1$	$-\boldsymbol{i}_1$	1	$-\boldsymbol{i}_3$	$\boldsymbol{i}_2$	$\boldsymbol{i}_1$	-1	$\boldsymbol{i}_3$	$-\boldsymbol{i}_3$
$-\boldsymbol{i}_2$	$-\boldsymbol{i}_2$	$\boldsymbol{i}_3$	1	$-\boldsymbol{i}_1$	$\boldsymbol{i}_2$	$-\boldsymbol{i}_3$	-1	$\boldsymbol{i}_1$
$-\boldsymbol{i}_3$	$-\boldsymbol{i}_3$	$-\boldsymbol{i}_2$	$\boldsymbol{i}_1$	1	$\boldsymbol{i}_3$	$\boldsymbol{i}_2$	$-\boldsymbol{i}_1$	-1

The multiplicative neutral is the scalar 1 and it has the additive inverse scalar factor -1.
Linearity with four real mixed dimensions, as the other four is the additive inverse of these.
In all; two independent generalised plane ***directions*** as specified (6.126), see Figure 6.1,t and (E XI.De.6.). These planes imply a mutual perpendicular third plane.
In the tradition, these planes have two Cartesian $\mathbb{R}$ dimensions each. These are equivalent to one complex number $\mathbb{C}$ dimension for each complex plane. Two planes $\mathbb{R}^4 \sim \mathbb{C}^3$ imply three planes in a strange mixed 3-dimensional way. Instead, we here try quaternions $\mathbb{H}$.

6.4.3.2. Quaternions ℍ

From this basis group (6.130) we form a linear space of multivectors over the real field $\mathbb{R}$.
This we as Hamilton call **quaternions**[311]

(6.131) $$Q = q_0 + q_k\boldsymbol{i}_k = q_0 1 + q_1\boldsymbol{i}_1 + q_2\boldsymbol{i}_2 + q_3\boldsymbol{i}_3 \in \mathbb{H}, \qquad \text{where } \forall q_0, q_k \in \mathbb{R} \text{ and } k = 1,2,3.$$

The reversed orientated (or *Clifford conjugated*) of a **quaternion *direction*** we define as

(6.132) $$\tilde{Q} = q_0 - q_k\boldsymbol{i}_k \in \mathbb{H}. \qquad \text{Here in 3-space with the even algebra } \mathbb{H} \sim \mathcal{G}_{0,2}(\mathbb{R}) \text{ we use } Q^\dagger = \tilde{Q}.$$

The auto product square is $Q^2 = QQ = q_0^2 - q_1^2 - q_2^2 - q_3^2 \in \mathbb{R}$, and the quaternion norm is

(6.133) $$|Q|^2 = Q\tilde{Q} = QQ^\dagger = q_0^2 + q_1^2 + q_2^2 + q_3^2 > 0.$$

[311] Pseudonymous Hamilton named the **quaternion** basis $\boldsymbol{i} \equiv \boldsymbol{i}_1$, $\boldsymbol{j} \equiv \boldsymbol{i}_2$, $\boldsymbol{k} \equiv \boldsymbol{i}_3$, where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$, as (6.124).
Throughout history, Hamilton's names $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are common for his quaternion basis 'vectors'. Hamilton was the first to use the term 'vector' as a mathematical-geometrical concept. The object names $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ also have been used for a 1-vector basis in the Gips 'vector' tradition where $\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}$. E.g. for waves, $\boldsymbol{k}$ had been used for the autonomous wavevector as a 1-vector.
Maxwell used the quaternion idea to develop his electromagnetic equations, but Lord Kelvin then Gips reformulated it, and the chiral information was lost in the transcendental, and the unproductive idea of axial vectors was born. The importance of the ***chiral direction*** was first problematised by I. Kant 1768 [11]p.361-372, see note[90]. – Therefore, we will use geometric algebra.