6.4.1.4. Plane Subjects in Euclidean $\mathfrak{Z}$ Space Clifford Algebra $\mathcal{G}_{3,0}$

For any plane substance defined by two orthonormal $\left(\boldsymbol{\sigma}_k{\cdot}\boldsymbol{\sigma}_j{=}\delta_{kj}\right)$ basis 1-vector objects $\{\boldsymbol{\sigma}_k,\boldsymbol{\sigma}_j\}$ in a Euclidean $\mathfrak{Z}$-space we write the full product of these basis 1-vectors

(6.101) $\boldsymbol{\sigma}_{kj}=\boldsymbol{\sigma}_k\boldsymbol{\sigma}_j=\langle\boldsymbol{\sigma}_k\boldsymbol{\sigma}_j\rangle_0+\langle\boldsymbol{\sigma}_k\boldsymbol{\sigma}_j\rangle_2=\boldsymbol{\sigma}_k{\cdot}\boldsymbol{\sigma}_j+\boldsymbol{\sigma}_k{\wedge}\boldsymbol{\sigma}_j.$ Where special for

(6.102) $k{\neq}j\ \Rightarrow\ \boldsymbol{\sigma}_{kj}{=}\boldsymbol{\sigma}_k{\wedge}\boldsymbol{\sigma}_j{=}\ \boldsymbol{i}\boldsymbol{\sigma}_l{=}\boldsymbol{i}_l,$ where $k{\neq}\ l\ {\neq}j,$ and alternative $k{=}j\Rightarrow\ \boldsymbol{\sigma}_{kk}{=}\boldsymbol{\sigma}_k\boldsymbol{\sigma}_k{=}1.$

From this, we first write the matrix form where the elements are the separated products of $\boldsymbol{\sigma}_k\boldsymbol{\sigma}_j$,

(6.103) $$\boldsymbol{\sigma}_{kj}\ \leftrightarrow\begin{bmatrix}\boldsymbol{\sigma}_k\boldsymbol{\sigma}_k & \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k\\ \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j & \boldsymbol{\sigma}_j\boldsymbol{\sigma}_j\end{bmatrix}=\begin{bmatrix}\boldsymbol{\sigma}_k{\cdot}\boldsymbol{\sigma}_k & \boldsymbol{\sigma}_j{\cdot}\boldsymbol{\sigma}_k\\ \boldsymbol{\sigma}_k{\cdot}\boldsymbol{\sigma}_j & \boldsymbol{\sigma}_j{\cdot}\boldsymbol{\sigma}_j\end{bmatrix}_0+\begin{bmatrix}\boldsymbol{\sigma}_k{\wedge}\boldsymbol{\sigma}_k & \boldsymbol{\sigma}_j{\wedge}\boldsymbol{\sigma}_k\\ \boldsymbol{\sigma}_k{\wedge}\boldsymbol{\sigma}_j & \boldsymbol{\sigma}_j{\wedge}\boldsymbol{\sigma}_j\end{bmatrix}_2\ =\mathbf{1}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\boldsymbol{\sigma}_k{\wedge}\boldsymbol{\sigma}_j\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

(6.104) $$\boldsymbol{\sigma}_{kj}\ \leftrightarrow\ \begin{pmatrix}1&-\boldsymbol{i}_l\\ \boldsymbol{i}_l&1\end{pmatrix}\ =\ \mathbf{1}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\boldsymbol{i}_l\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

For the three perpendicular planes in $\mathfrak{Z}$-space, we choose matrix indices as column sequential cyclic order by colour in the following framework ⟶ ⟶ $\begin{Bmatrix}j=1,3,2\\k=2,1,3\\l=3,2,1\end{Bmatrix}$
This matrix structure gives the foundation for the Pauli matrices (see below)
We see the equivalent to a basis of an *even* multivector algebra $\mathcal{G}^+_{3,0}$ for one plane, e.g., for indices l

(6.105) $A\rightarrow\ \langle A\rangle^+_{0,2}=\langle A\rangle_0+\langle A\rangle_2\ =\ \alpha_{kj}\mathbf{1}+\left(\beta_{kj}\right)\boldsymbol{\sigma}_{kj}=\ \alpha_l+(\beta_l)\boldsymbol{i}_l,$ where $\alpha_l,\beta_l\ \in\mathbb{R}$.[305]

To fully understand this rotor as (6.101) for the plane concept we recall from the original definition (5.56)-(5.60), that a reversion changes the orientation of the rotor ***direction***.
In general, the reverse order in geometric algebra is expressed as $(AB)^\dagger=B^\dagger A^\dagger$, from this it is obvious that for 1-vectors $\mathbf{a}^\dagger=\mathbf{a}$ or $\langle A\rangle_1^\dagger{=}\langle A\rangle_1$ because there is essentially only one 1-vector in $\langle A\rangle_1$. Scalars has no ***direction*** (***pqg***-0), therefore $\langle A\rangle_0^\dagger=\langle A\rangle_0$ too.

For the simplest product, we have $(\mathbf{ac})^\dagger=\mathbf{c}^\dagger\mathbf{a}^\dagger=\mathbf{ca}$. For the reverse of (6.105) we have

(6.106) $A^\dagger\rightarrow\ \langle A\rangle^{+\dagger}_{0,2}\ =\langle A\rangle_0-\langle A\rangle_2\ \sim\ \alpha_{kj}1-\left(\beta_{kj}\right)\boldsymbol{\sigma}_{kj}=\ \alpha_l-(\beta_l)\boldsymbol{i}_l.$

From this complicated matrix concept idea , we get

(6.107) $\langle A\rangle^+_{0,2}\langle A\rangle^{+\dagger}_{0,2}\ =\langle A\rangle^2_0+\langle A\rangle^2_2\ =\alpha^2+\beta^2,$ and $\boldsymbol{\sigma}_{kj}{}^2{=}-1$ for $k{\neq}j,\ {=}1{,}2{,}3.$

We repeat the 1-rotor concept from (6.77) combined with (6.71), (6.72) ← (5.191), (5.192)

(6.108) $U=\widetilde{\mathbf{uv}}=\mathbf{vu}=\mathbf{v}{\cdot}\mathbf{u}+\mathbf{v}{\wedge}\mathbf{u}=e^{+½\boldsymbol{\theta}}\ =\ U_{\phi\widehat{\boldsymbol{\omega}}}{=}\ e^{+\boldsymbol{i}\widehat{\boldsymbol{\omega}}\phi}\ =\ 1\cos\phi+\boldsymbol{i}\widehat{\boldsymbol{\omega}}\sin\phi$

(6.109) $U^\dagger{=}\widetilde{U}=\widetilde{\mathbf{uv}}=\mathbf{vu}=\mathbf{v}{\cdot}\mathbf{u}-\mathbf{v}{\wedge}\mathbf{u}=e^{-½\boldsymbol{\theta}}\ =\ \widetilde{U}_{\phi\widehat{\boldsymbol{\omega}}}{=}\ e^{-\boldsymbol{i}\widehat{\boldsymbol{\omega}}\phi}\ =\ 1\cos\phi-\boldsymbol{i}\widehat{\boldsymbol{\omega}}\sin\phi$

where we have the rotor angle between two 1-vectors $\phi{=}\sphericalangle(\mathbf{u,v})$, and the bivector $½\boldsymbol{\theta}=\boldsymbol{i}\widehat{\boldsymbol{\omega}}\phi$ as an argument in the multivector exponential function $e^{½\boldsymbol{\theta}}$. We have unitarity

(6.110) $U\widetilde{U}\ =\ UU^\dagger=\cos^2\phi+\sin^2\phi\ \ =1.$

From the orthogonal basis matrix structure (6.103) where we choose $\widehat{\boldsymbol{\omega}}=\widehat{\boldsymbol{\omega}}_l\equiv\boldsymbol{\sigma}_l{=}-\boldsymbol{i}\boldsymbol{\sigma}_{kj}$

(6.111) $\boldsymbol{\sigma}_{kj}\ \leftrightarrow\ 1\begin{bmatrix}1_k&0_l\\0_l&1_j\end{bmatrix}+\boldsymbol{i}\widehat{\boldsymbol{\omega}}_l\begin{bmatrix}0_k&-1_l\\1_l&0_j\end{bmatrix},$ where $l{\neq}j{\neq}k{\neq}l$, and $j,k,l{=}1{,}2{,}3$ as (6.101).

For the trigonometric functions from the rotation angle for the real $2{\times}2$ rotation matrix we have

(6.112) $\phi\longrightarrow\ \cos\phi\begin{bmatrix}1&0\\0&1\end{bmatrix}+\sin\phi\begin{bmatrix}0&-1\\1&0\end{bmatrix}\ =\ \begin{bmatrix}\cos\phi&-\sin\phi\\ \sin\phi&\cos\phi\end{bmatrix},$ (without specified ***direction***).

The set of these functions is called the special orthogonal group $SO(2)$ mentioned (1.54), with the determinant $\begin{vmatrix}\cos\phi&-\sin\phi\\ \sin\phi&\cos\phi\end{vmatrix}=1,$ equivalent to (6.110)
When we instead use the complex number plane picture for the circle group $\mathbb{T}\subset\mathbb{C}$, from I. (1.51)

(6.113) $\mathbb{R}\rightarrow\mathbb{T}:\ \phi\rightarrow\ \ e^{i\phi}\ \ =\cos\phi+i\sin\phi$

this group is isomorph equivalent to the unitary $U(1)$ where $e^{i\phi_1}e^{i\phi_2}=e^{i(\phi_1+\phi_2)}$ is in one plane.

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[305] Here we do not use Einstein sum over double indexes, $\cancel{\sum_{jk}}\left(\beta_{kj}\right)\boldsymbol{\sigma}_{kj}$. The sum is valid, but not the essence for one plane.

In this the complex numbers pure imaginary unit $i\in\mathbb{C}$, with $i^2{=}-1$, is indeed not a bivector narrative for the intuition, but intentionally abstract as in the a priory transcendental tradition.
Anyway, the group $U(1)$ structure is isomorph equivalent to the multivector concept of a 1-rotor.

(6.114) $u\!:\ \phi\rightarrow U_{\phi\widehat{\boldsymbol{\omega}}}=\ \ e^{\boldsymbol{i}\widehat{\boldsymbol{\omega}}\phi}=1\cos\phi+\boldsymbol{i}\widehat{\boldsymbol{\omega}}\sin\phi,$ (In the transversal plane ***direction*** of $\widehat{\boldsymbol{\omega}}$),

as one real parameter multivector function for the angular rotation in a ***directional*** plane $\boldsymbol{i}\widehat{\boldsymbol{\omega}}$.
These elements as in the group $U(1)$ give subjects in Geometric Algebra that is even $\mathcal{G}^+_{3,0}$ in the total $\mathcal{G}_{3,0}$ as rotor $\langle A\rangle^+_{0,2}{=}U_{\phi\widehat{\boldsymbol{\omega}}}{=}U_\phi{=}\mathbf{vu}{\perp}\widehat{\boldsymbol{\omega}}$, around an object $\widehat{\boldsymbol{\omega}}$ in $\mathfrak{Z}$-space of physics.[306]

6.4.2. The Pauli Basis for combined direction structures of $\mathfrak{Z}$-space

We again choose the standard 1-vector basis $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ for the geometric algebra $\mathcal{G}_{3,0}(\mathbb{R})$.
Do we choose the ***direction*** so $\widehat{\boldsymbol{\omega}}=\boldsymbol{\sigma}_3$ we can rewrite the $\langle A\rangle_2$ unit part of (6.103) and (6.111)

(6.115) $$\langle A\rangle_2\leftrightarrow\boldsymbol{\sigma}_2{\wedge}\boldsymbol{\sigma}_1\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\boldsymbol{\sigma}_{21}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\boldsymbol{i}\widehat{\boldsymbol{\omega}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\ \boldsymbol{i}\boldsymbol{\sigma}_3\begin{pmatrix}0&-1\\1&0\end{pmatrix}\leftrightarrow\ \begin{cases}-\boldsymbol{i}\boldsymbol{\sigma}_3=\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2=-\boldsymbol{i}_3,\\+\boldsymbol{i}\boldsymbol{\sigma}_3=\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1=+\boldsymbol{i}_3.\end{cases}$$

We see (as seen so often before) that the transversal ***pqg***-2 rotation orientation around a ***pqg***-1 ***direction*** $\widehat{\boldsymbol{\omega}}$ has two states. As we may know[307] there are just *three* ***categorical*** independent 1-vector ***directions*** in $\mathfrak{Z}$-space of physics, locally represented by different objects $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$.

6.4.2.2. The Pauli Matrices as generator operators

Wolfgang Pauli expressed these differences and dependency by three fundamental matrices:

(6.116) $\hat{\sigma}_1{=}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\hat{\sigma}_2{=}i\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad\hat{\sigma}_3{=}\begin{pmatrix}1&0\\0&-1\end{pmatrix},$ and the *neutral Identity* $\hat{\sigma}_0{=}\begin{pmatrix}1&0\\0&1\end{pmatrix}.$

The identity matrix expresses that unitary structure by $\hat{\sigma}_1^{\,2}=\hat{\sigma}_2^{\,2}=\hat{\sigma}_3^{\,2}=\ \hat{\sigma}_0^{\,2}=\ \hat{\sigma}_0.$
And $i=\sqrt{-1}$, i.e., $i^2{=}-1$ is a ***quality*** multiplication operation, that when used two times reverse the orientation of a ***direction*** with consequence two stats of orientation (±)
From this, we construct the closed Pauli group that consists of $2^4{=}16$ elements

(6.117) $$\left\{\pm\hat{\sigma}_0,\ \ \pm\hat{\sigma}_1,\ \pm\hat{\sigma}_2,\ \pm\hat{\sigma}_3,\quad\pm i\hat{\sigma}_1{=}\begin{cases}\hat{\sigma}_2\hat{\sigma}_3\\ \hat{\sigma}_3\hat{\sigma}_2\end{cases},\quad\pm i\hat{\sigma}_2{=}\begin{cases}\hat{\sigma}_3\hat{\sigma}_1\\ \hat{\sigma}_1\hat{\sigma}_3\end{cases},\quad\pm i\hat{\sigma}_3{=}\begin{cases}\hat{\sigma}_1\hat{\sigma}_2\\ \hat{\sigma}_2\hat{\sigma}_1\end{cases},\quad\pm i\hat{\sigma}_0\right\}.$$

The three generating multiplication matrix operators $\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3$ of the Pauli group produce $2^3{=}\,8$ different ***direction qualities*** with double ± orientation in physics.
A closer look at this closed group shows a closed subgroup we call the *Lifted Pauli Group*

(6.118) $$\left\{\pm\hat{\sigma}_0,\ \pm i\hat{\sigma}_1,\ \pm i\hat{\sigma}_2,\ \pm i\hat{\sigma}_3\right\}\quad=\left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \pm\begin{pmatrix}0&i\\i&0\end{pmatrix},\ \pm\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ \pm\begin{pmatrix}i&0\\0&-i\end{pmatrix}\right\}.$$

With $2^3{=}\,8$ elements representing $2^2{=}\,4$ ***direction qualities*** of double ± orientation.
We return to the more intuitive picture of multivectors in Geometric Algebra.

6.4.2.3. The Pauli Basis Generated from 1-vectors

From the dextral orthonormal basis $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ of 1-vectors for the geometric algebra $\mathcal{G}_{3,0}(\mathbb{R})$ for $\mathfrak{Z}$-space we have the mixed basis of $2^3{=}\,8$ ***direction qualities*** that we call a *Pauli basis*[308]

(6.119) $\{1,\ \ \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3,\ \ \boldsymbol{i}\boldsymbol{\sigma}_1=\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2,\ \ \boldsymbol{i}\boldsymbol{\sigma}_2=\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3,\ \ \boldsymbol{i}\boldsymbol{\sigma}_3=\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1,\quad\boldsymbol{i}\coloneqq\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\}$

Of these 8 ***direction qualities***, there are one ***pqg***-0, three ***pqg***-1, three ***pqg***-2 planes, and one ***pqg***-3 ***quality*** for volume. The first is a scalar and has as such ***no direction*** in $\mathfrak{Z}$-space of physics.
All of these eight ***qualities*** can ***quantitively*** be scaled with both positive and negative reals $\mathbb{R}$, so, we do not need the ± f or orientations as in (6.117) for this generating mixed basis elements to be complete in the geometric algebra $\mathcal{G}_{3,0}(\mathbb{R})$. (In all with $2^4{=}16$ generators as (6.117).)
For every multivector in $\mathcal{G}_{3,0}(\mathbb{R})$ from (6.59), we have a linear combination

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[306] In physics, there is something outside $\langle A\rangle_1{\sim}\ \widehat{\boldsymbol{\omega}}$, this is the transversal rotation plane $\langle A\rangle_2{\sim}\ \boldsymbol{i}\widehat{\boldsymbol{\omega}}$, as an impact of $\langle A\rangle_3{\sim}\ \boldsymbol{i}$.

[307] Immanuel Kant was the first to reason this[178] by the Kepler-Newton square law from Descartes's original extension idea.

[308] David Hestenes call the algebra for this group the *Pauli algebra* [6]p.16, §'6 The Algebra of Space' 1966\2015.