6.3.6. Rotation of Multivectors

After this, we rotate a bivector (5.58) $\mathbf{B} = \mathbf{c}\wedge\mathbf{a} = \frac{1}{2}(\mathbf{ca} - \mathbf{ac})$ by rotating each 1-vector

(6.86) $$\underline{\mathcal{R}}\mathbf{B} = U\mathbf{B}U^\dagger = U(\mathbf{c}\wedge\mathbf{a})U^\dagger = \frac{1}{2}U(\mathbf{ca} - \mathbf{ac})U^\dagger = \frac{1}{2}(U\mathbf{c}U^\dagger U\mathbf{a}U^\dagger - U\mathbf{a}U^\dagger U\mathbf{c}U^\dagger) = \mathbf{c}'\wedge\mathbf{a}' = \mathbf{B}'.$$

here we used that $U^\dagger U = 1$.[301] That is simple, $\underline{\mathcal{R}}$ rotates a bivector to a bivector just as a 1-vector rotates to a 1-vector. The unit ***pqg***-3 trivector $\boldsymbol{i}$ as chiral volume pseudoscalar $\langle A\rangle_3$ commute with all terms in the $\mathcal{G}_3(\mathbb{R})$ algebra therefor

(6.87) $$U\boldsymbol{i}U^\dagger = UU^\dagger\boldsymbol{i} = \boldsymbol{i}$$

$\boldsymbol{i}$ is rotation invariant. The scalar $\langle A\rangle_0$ is of cause too rotation invariant

(6.88) $$\underline{\mathcal{R}}\langle A\rangle_0 = U\langle A\rangle_0 U^\dagger = UU^\dagger\langle A\rangle_0 = \langle A\rangle_0$$

We here see that rotation $\underline{\mathcal{R}}$ preserve ***grades*** in the $\mathcal{G}_3(\mathbb{R})$ geometric algebra

(6.89) $$\underline{\mathcal{R}}\langle A\rangle_r = U\langle A\rangle_r U^\dagger = \langle A\rangle_r'$$

All these ***grades*** are connected in rotation by e.g.

(6.90) $$\left(\boldsymbol{i}\underline{\mathcal{R}}\mathbf{B} = \boldsymbol{i}U\mathbf{B}U^\dagger = U\boldsymbol{i}\mathbf{B}U^\dagger = \boldsymbol{i}\mathbf{B}' \quad\text{and}\quad \boldsymbol{i}\mathbf{B} = -\mathbf{b}\right) \Leftrightarrow \underline{\mathcal{R}}\mathbf{b} = U\mathbf{b}U^\dagger = \mathbf{b}'.$$

We then conclude that all multivectors constructed of a polynomial of all ***grades***

(6.91) $$A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \langle A\rangle_3 + \cdots$$

rotates in the same manner without mixing the grades.
In the traditional matrix representation in frame coordinates, this is called an *orthogonal rotation*.

6.3.7. Framing a Field for Geometric Algebra in $\mathfrak{Z}$-space

Giving a dextral (righthanded) orthonormal basis $\{\mathbf{e}_j, j{=}1{,}2{,}3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as a founding object for a Cartesian coordinate system[302] for a straight-line field structure in $\mathfrak{Z}$-space. We can obtain any local orthonormal (standard) frame by an orthogonal rotation in the canonical form

(6.92) $$\boldsymbol{\sigma}_j = U\mathbf{e}_j U^\dagger$$

This operation[303] is a mapping of the frame

(6.93) $$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \longrightarrow \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}.$$

The inverse mapping

(6.94) $$\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\} \longrightarrow \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\},$$

with the inverse operation

(6.95) $$\mathbf{e}_k = U^\dagger\boldsymbol{\sigma}_k U,$$

due to $UU^\dagger = U^\dagger U = 1$.
Alternatively, the local frame can be expressed by a rotation matrix

(6.96) $$\mathbf{e}_k = \alpha_{j,k}\boldsymbol{\sigma}_j \qquad = \textstyle\sum_j \alpha_{j,k}\boldsymbol{\sigma}_j$$

The matrix elements can be solved as a scalar function of U

(6.97) $$\alpha_{k,j} = \boldsymbol{\sigma}_k\cdot\mathbf{e}_j = (U\mathbf{e}_k U^\dagger)\cdot\mathbf{e}_j = \langle U\mathbf{e}_k U^\dagger\mathbf{e}_j\rangle_0.$$

or

(6.98) $$\alpha_{j,k} = \boldsymbol{\sigma}_j\cdot\mathbf{e}_k = \boldsymbol{\sigma}_j\cdot(U^\dagger\boldsymbol{\sigma}_k U) = \langle\boldsymbol{\sigma}_j U^\dagger\boldsymbol{\sigma}_k U\rangle_0.$$

To do this the reader can study this further in the literature, e.g., [10]p.286ff.

[301] This deduction is inspired by [18] p48.
[302] Where $\mathbf{e}_2\perp\mathbf{e}_1$, $\mathbf{e}_3\perp\mathbf{e}_2$, $\mathbf{e}_1\perp\mathbf{e}_3$, and $|\mathbf{e}_1| = |\mathbf{e}_2| = |\mathbf{e}_3| = 1$, as orthonormal $\mathbf{e}_j\cdot\mathbf{e}_k = \frac{1}{2}(\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_j\mathbf{e}_k) = \delta_{jk}$, $j,k = 1,2,3$.
And where translation invariance is presumed obvious as well as we have Galileo translation invariance over time.
Different local points P in $\mathfrak{Z}$ space relative to an origo O for the basis $\{\mathrm{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a problem we already know.
[303] This is written conversely to [5]p.23-24 and [10]3.31p.286.

6.4. The Geometric Clifford Algebra

Due to Hestenes, Clifford called his multiplication algebra for Geometric Algebra, so here we call the real Clifford algebra for Geometric Algebra with the terms $\mathcal{G}_n = \mathcal{G}_n(\mathbb{R}) = \mathcal{G}(V_n,\mathbb{R}) \sim C\ell_n(V,\mathbb{R})$.
This type of linear algebra can be equipped with different types of basis vectors. E.g.:
$\mathcal{G}_3(\mathbb{R})$ has the intuit object standard basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ of 1-vectors, where $\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_3^2 = 1$.
Further the dual basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\} = \{\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_1\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\}$ of bivectors, where $\boldsymbol{i}_1^2 = \boldsymbol{i}_2^2 = \boldsymbol{i}_3^2 = -1$,
is intuited as a subject orthonormal basis for the substance *idea of planes* in $\mathfrak{Z}$-space of physics.

6.4.1.1. The Quadratic Form in general

It is now time to expand the metric quadratic form from (5.37) and (5.42) $\mathbf{v}^2 = \boldsymbol{Q}(\mathbf{v})\,\epsilon_A$,
Where we have the possible signatures $\epsilon_A = 1, 0, -1$. We can construct a linear space of dimension $n = \dim(V_n)$ where the generating 1-vector spaces V_p has positive signed quadrats and the rest of dimensions $V_{q=n-p}$ has negative signed quadrats.
We combined both by addition to a linear space $V_n = V_p \oplus V_q$.
Multiplication of these 1-vectors $v_k \in V_n$, $k{=}1{,}2{,}\dots{,}n$, forming polynomial multivectors generating the linear spaces of the geometric algebra $\mathcal{G}_{p,q} \leftarrow \mathcal{G}_n$ where $n{=}p{+}q$, we define the quadratic form

(6.99) $$Q(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2 \quad \in \mathbb{R}$$

This geometric algebra $\mathcal{G}_{p,q}(\mathbb{R}) = \mathcal{G}(V_n,\mathbb{R})$ is equal to a Clifford algebra $C\ell_{p,q}\left(\forall v{\in}V_n, \mathbb{R}, Q(v)\right)$.

6.4.1.2. The Clifford Algebra for Complex Numbers

Short, the quadratic form also works for $\mathbb{C}$: $Q(z) = z_1^2 + z_2^2 + \cdots + z_n^2$, then we e.g., write
$C\ell_0(\mathbb{C}) \sim \mathbb{C}$, $\quad C\ell_1(\mathbb{C}) \sim ([\mathbb{C}], \mathbb{C})$, $\quad C\ell_2(\mathbb{C}) \sim \left(\begin{bmatrix}\mathbb{C} & \mathbb{C}\\ \mathbb{C} & \mathbb{C}\end{bmatrix}, \mathbb{C}\right)$, …
We will not go further into this right here, but history is rich in this. Anyway:
The complex number $\mathbb{C}$ is good for the complex plane idea, such as the transversal plane concept.
But: We here stick to the real field $\mathbb{R}$ for a general geometric algebra $\mathcal{G}_{p,q} = \mathcal{G}_{p,q}(\mathbb{R}) = \mathcal{G}(V_n,\mathbb{R})$.[304]

6.4.1.3. The simple Euclidean Plane Geometric Clifford Algebra $\mathcal{G}_{2,0}$

A plane concept $\mathfrak{P}$ we traditional span by the Cartesian coordinate system from the orthonormal basis set $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$ as a 2-dimensional 1-vector space $(V_2,\mathbb{R})$ the geometric algebra for this is $\mathcal{G}_{2,0}$ and for this, we have the $2^2 = 4$-dimensional linear mixed ***grades***.
The multivector for this has the ***grade*** structure $A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2$ in $\mathcal{G}_{2,0}(\mathbb{R})$.
We name an orthonormal basis $\{1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_{21} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\}$ for this, which have the group multiplication structure Table 6.1:

Table 6.1 Multiplication basis for $\mathcal{G}_{2,0}$.

left* \right	1	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_{21}$
1	1	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_{21}$
$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_1$	1	$-\boldsymbol{\sigma}_{21}$	$-\boldsymbol{\sigma}_2$
$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_{21}$	1	$\boldsymbol{\sigma}_1$
$\boldsymbol{\sigma}_{21}$	$\boldsymbol{\sigma}_{21}$	$\boldsymbol{\sigma}_2$	$-\boldsymbol{\sigma}_1$	-1

Multiplication of all elements with -1 closes the multiplication group for this plane $\mathcal{G}_{2,0} = \mathcal{G}_2(\mathbb{R})$ algebra.
The $\langle A\rangle_2$ (***pqg***-2) unit bivector $\boldsymbol{\sigma}_{21} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ squares to $\boldsymbol{\sigma}_{21}{}^2 = -1$, reverses $\boldsymbol{\sigma}_{12} = -\boldsymbol{\sigma}_{21}$ and anticommute with all 1-vectors in its own plane $\boldsymbol{\sigma}_{21}\mathbf{x} = -\mathbf{x}\boldsymbol{\sigma}_{21}$.
From this, we span the full multivector algebra $A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2$ for $\mathcal{G}_{2,0}$.
First, the scalar $\langle A\rangle_0 = \alpha 1$, where $\alpha \in \mathbb{R}$ and the general bivector $\langle A\rangle_2 = \beta_3\boldsymbol{\sigma}_{21}$, where $\beta_3 \in \mathbb{R}$.
Then we have that any 1-vector $\langle A\rangle_1$ is expressed in the odd algebra $\mathcal{G}_{2,0}^-$ as

(6.100) $$\langle A\rangle_1 = \quad \mathbf{x} = x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 \quad \leftrightarrow \quad (x_1, x_2)\begin{pmatrix}\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_2\end{pmatrix} \quad \text{in a matrix formulation.}$$

For this Cartesian plane, we have the quadratic metric $\mathbf{xx} = \mathbf{x}^2 = x_1^2 + x_2^2$ and general for a Euclidean space $\mathbf{x}^2 = x_k x_k = \sum_k x_k^2$ with the orthonormal basis $\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_j = \delta_{kj}$.
To enrich the plane concept with the complex numbers $\mathbb{C}$, will be awkward when it comes to $\mathfrak{Z}$-space with $\mathcal{G}_{3,0}$, instead we will stick to the form $\langle A\rangle_0 + \langle A\rangle_2$ for plane spinors of $\mathcal{G}_{0,2}(\mathbb{R}) \sim \mathcal{G}_{3,0}^+$.

[304] Just as David Hestenes [6], [10], [5], [33], etc. uses the real field in his new foundation of geometric algebra for physics.