6.3.5. Multiplication Combination of Rotors

6.3.5.1. The Unitary group $U(1)$ for Plane Combination of Rotors

The unitary plane 1-spinor rotor U belong to the simple unitary group $U(1)$. We have its elements as complex functions $u\colon\ \phi \to e^{i\phi} \in \mathbb{C}$ of one parameter $\phi \in \mathbb{R}$, which is commutative in multiplication inherited from the additive commutative of the parameter, due to the rule from (1.55): $u(\phi_1)\cdot u(\phi_2) = u(\phi_1+\phi_2) \sim e^{i\phi_1}e^{i\phi_2} = e^{i(\phi_1+\phi_2)}$, thus $U(1)$ is an Abelian, unitary since $u^*u=1$, as well as a group, since u and u^* are each other's multiplicative inverse, by the neutral element. This one parameter group is cyclic identical as (1.56): $u(2\pi n) = e^{2\pi n} = e^0 = 1$, for $\forall n \in \mathbb{Z}$. We will call these Abelian elements isomorph with *1-rotors* of the geometric algebra.
The circular rotation substance has a ***primary quality of even grades*** 0 and 2, given by $U(1)$.
The elements in the group $U(1)$, $u\colon\ \phi \to U_\phi$, give subjects in the geometric algebra $\mathcal{G}_3^+(\mathbb{R})$ by a rotor $\langle A\rangle_{0,2}^+ = U_\phi = \mathbf{vu}$, with the one parameter $\phi = 2\measuredangle(\mathbf{v},\mathbf{u})$, for 𝔍-space of physics.
The mandatory issue here is that the $U(1)$ multiplicative Abelian group is substantially connected to the plane idea. In 𝔍-space idea, we are forced to look at an idea of a bivector $\boldsymbol{i}\hat{\omega}$ for the transversal plane ***direction*** to implicitly to a 1-vector ***direction*** $\hat{\omega}$ with the plane operator

(6.77) $$U_{\phi\hat{\omega}} = e^{\boldsymbol{i}\hat{\omega}\frac{1}{2}\phi},\quad \text{with the transversal plane Abelian rule } U_{(\phi_1+\phi_2)\hat{\omega}} = U_{\phi_1\hat{\omega}}U_{\phi_2\hat{\omega}} = U_{\phi_2\hat{\omega}}U_{\phi_1\hat{\omega}}.$$

6.3.5.2. Multiplication Combination of *Direction* Different 1-rotors in 𝔍-space

Different rotation ***directions*** can be combined. First, we look at different 1-vector ***directions*** $\hat{\omega}$, $\hat{\omega}_1$ and $\hat{\omega}_2$, thus $\hat{\omega}_1 \neq \hat{\omega}_2$, geometrically $\hat{\omega}_1 \nparallel \hat{\omega}_2$. From these we have plane ***directions*** $\boldsymbol{i}\hat{\omega}_1$ and $\boldsymbol{i}\hat{\omega}_2$, in which the angular 1-rotors have different ***pqg***-2 ***directions***

(6.78) $$U_1 = U_{\theta_1\hat{\omega}_1} = e^{\boldsymbol{i}\hat{\omega}_1\frac{1}{2}\theta_1},\quad \text{and}\quad U_2 = U_{\theta_2\hat{\omega}_2} = e^{\boldsymbol{i}\hat{\omega}_2\frac{1}{2}\theta_2}\quad \in \mathcal{G}_3^+(\mathbb{R}) \subset \mathcal{G}_3(\mathbb{R}).$$

These two subject 1-rotors we intuit as products of three unit 1-vector objects $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and write $U_1 = \mathbf{vu} \to \odot$ and $U_2 = \mathbf{wv} \to \odot$ displayed in Figure 6.13. From these, we make the rotor product

(6.79) $$U_3 = U_2U_1 = U_2U_1 = (\mathbf{wv})(\mathbf{vu}) = \mathbf{wu} \quad \to \odot$$

We see that these two circular rotors U_1 and U_2 intersect along the 1-vector object $\mathbf{v}$, representing the a priori intuition for the interaction of the product of the two rotors.

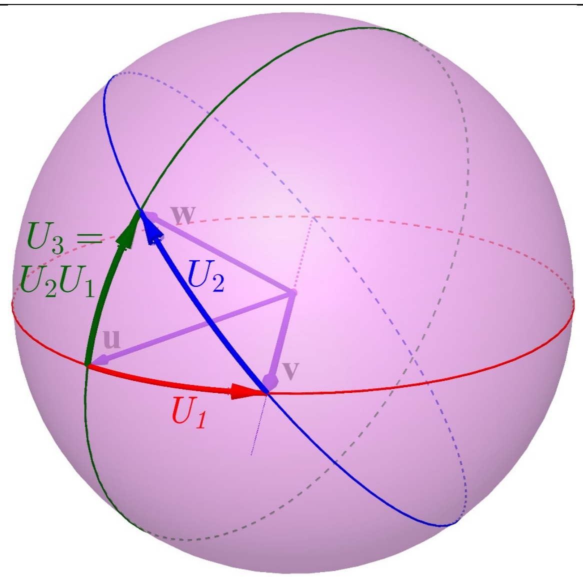

Figure 6.13 Rotor multiplication in 𝔍 space gives $U_3 = U_2U_1 = (\mathbf{wv})(\mathbf{vu}) = \mathbf{wu}$, as an arc object on the unit sphere that result in a new circular plane ***direction*** $\odot$.
$\mathbf{v}$ represents the intersecting 1-vector interaction.

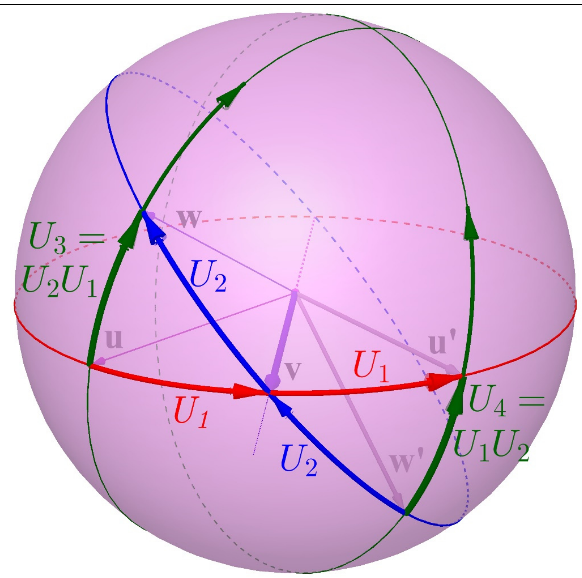

Figure 6.14 Non-commuting rotors $U_2U_1 \not\approx U_1U_2$
$U_4 = U_1U_2 = (\mathbf{u'v})(\mathbf{vw'}) = \mathbf{u'w'} = (\mathbf{vuvv})(\mathbf{vvwv}) = \mathbf{vuwv}$.
Different ***directions*** for these two resulting planes.
Four regular circular rotations $\mathcal{R}_k = U_k^2 = e^{\boldsymbol{i}\hat{\omega}_k\theta_k}$.

The rotors U_k are rotation-invariant in each their own circular plane (around in a whole circle $\odot$). Although circular rotors commute with others in that same plane, between externally different planes they do not commute: $U_2U_1 \not\approx U_1U_2$ as illustrated in Figure 6.14. To understand this, we auto-rotate the invariant rotors by using the symmetry of a unit 1-vector $\mathbf{vv} = \mathbf{v}^2 = 1$

(6.80) $$U_1 = \mathbf{vu} = \mathbf{vuvv} = \mathbf{u'v} \quad\text{and}\quad U_2 = \mathbf{wv} = \mathbf{vvwv} = \mathbf{vw'},$$

where we have reflected $\mathbf{u}$ and $\mathbf{w}$ in $\mathbf{v}$ and achieved the new objects $\mathbf{u'} = \mathbf{vuv}$ and $\mathbf{w'} = \mathbf{vwv}$.[298]

(6.81) $$U_4 = U_1U_2 = U_1U_2 = (\mathbf{u'v})(\mathbf{vw'}) = \mathbf{u'w'} \quad = (\mathbf{vuvv})(\mathbf{vvwv}) = \mathbf{vuwv}.$$

Each regular circular rotation we represent as a squared rotor $\mathcal{R}_k = U_k^2 = e^{\boldsymbol{i}\hat{\omega}_k\theta_k}$. (no k sum).
In Figure 6.14 we see in all four circular plane ***directions*** of these.

When acting on a physical ***entity*** as one object represented by a 1-vector ***direction*** we use the fundamental canonical form (sandwiching) for the operation $\mathcal{R}_k\mathbf{x} = U_k^2\mathbf{x} = U_k\mathbf{x}U_k^\dagger$.
For the combined rotation by multiplication of rotor operators we write

(6.82) $$\mathbf{x}_{1,2} = \mathcal{R}_{1,2}\mathbf{x} = U_{1,2}^2\mathbf{x} = U_{1,2}\mathbf{x}U_{1,2}^\dagger = (U_2U_1)\mathbf{x}(U_2U_1)^\dagger = U_3\mathbf{x}U_3^\dagger.$$

For the permutated rotor operator product U_2U_1 we write

(6.83) $$\mathbf{x}_{2,1} = \mathcal{R}_{2,1}\mathbf{x} = U_{2,1}^2\mathbf{x} = U_{2,1}\mathbf{x}U_{2,1}^\dagger = (U_1U_2)\mathbf{x}(U_1U_2)^\dagger = U_4\mathbf{x}U_4^\dagger \quad \not\approx \quad \mathbf{x}_{1,2} \quad !$$

The 2-rotors no longer belong to the Abelian group $U(1)$ but is rather an isomorph to $SU(2)$.[299]

6.3.5.3. Comment on the Ontology of *Directions* and Possibility of Location

Are we caught in a trap? We know, we have the idea of a 1-vector subject $\mathbf{v}$ as a translation invariant ***direction*** over all 𝔍-space. For intuition we define $\mathbf{v} \coloneqq \overrightarrow{AB}$ as an object of points we mark on a surface. The rotor objects we for intuition define as $U_1 \coloneqq \mathbf{vu}$ and $U_2 \coloneqq \mathbf{wv}$ have translation invariant subjects U_1 and U_2 with two independent arc-circular plane ***directions.***
These two ***pqg***-2 ***directions*** U_1 and U_2 intersects in a ***pqg***-1-vector subject $\mathbf{v}$ ***direction*** in 𝔍-space sphere volume. When we imagine the two arc-circular plane rotor subject U_1 and U_2 ***qualities***, we auto-inherit through the intersection the 1-vector $\mathbf{v}$ ***direction quality***. (as caught in a trap).
Dependent on the two arc ***quantities*** of U_1 and U_2 we inherit two extra independent ***pqg***-2 ***direction*** subjects defined by $U_3 \coloneqq U_2U_1$ and $U_4 \coloneqq U_1U_2$. Each of these two new arc-circular planes intersects the ***pqg***-1-vector $\mathbf{v}$ ***direction*** in just one point as a center of locality for this situation. This subject center of locality is translations invariant as all the subjects U_1, U_2, $\mathbf{v}$ and U_3 or U_4.
The situated center of the locality is caught in this trap we call a *2-rotor*. (isomorph with $SU(2)$)

- We as thinking observers are excluded from the external and can only point out some object mark-point as center on a chosen surface to symbolise the *locus situs* for this intuition. –

6.3.5.4. The Abstract Generalised Rotor Form

We have here above described the ***directional*** circular rotors (6.78) etc. as $U_j = e^{\boldsymbol{i}\hat{\omega}_j\frac{1}{2}\theta_j}$.
Each of these for $j \in \mathbb{N}$ describes its own independent plane ***pqg***-2 ***direction*** by the unit bivectors $\boldsymbol{i}_j = \boldsymbol{i}\hat{\omega}_j$, endowed with an angular parameter θ_j. Then each rotor is just written[300]

(6.84) $$U_j = e^{\boldsymbol{i}_j\frac{1}{2}\theta_j} \quad \in \mathcal{G}_3^+(\mathbb{R}) \subset \mathcal{G}_n(\mathbb{R}).$$

These rotor operators have ***directions*** that act on what stands to the right in the writing and change the ***direction*** of these operands. Be careful, the multivectors of the simple form

(6.85) $$U_\theta = e^{\boldsymbol{i}\frac{1}{2}\theta} \quad \notin \mathbb{C}, \qquad \text{but } \in \mathcal{G}_2(\mathbb{R}) \subset \mathcal{G}_3^+(\mathbb{R}) \subset \mathcal{G}_3(\mathbb{R}) \subset \cdots \subset \mathcal{G}_n(\mathbb{R}),$$

are indeed not complex scalars but represent the geometrical ***direction*** in physical 𝔍-space ⊂ 𝔖.

[298] § 5.4.2.1 II. 5.4.2.1 Reflection in a Geometric 1-vector.
[299] A practical physical example of this problem has been constructed in Rubrik's Cube, when we rotate in two or three planes (perpendicular) things get that complicated. The reader may look at [19] Figure.4.9-10p.164, or [10]Fig.3.5p.285.
[300] with the pure unitary complex number analogy $U = e^{-i\frac{1}{2}\theta} \in \mathbb{C}$ as a scalar without any physical ***direction***!