(6.45) $\quad \mathbf{aB} = \mathbf{a}(\mathbf{b}\wedge\mathbf{c}) \quad$ or $\quad \mathbf{Ba} = (\mathbf{b}\wedge\mathbf{c})\mathbf{a}$

These by (6.40), (6.43) and (6.38) consist of a 1-vector part $\langle\mathbf{aB}\rangle_1$ plus a trivector part $\langle\mathbf{aB}\rangle_3$.

6.2.5.4. The Simple Product of Three 1-vectors

Further by the associative law (5.38), we define a 3-multivector product of three 1-vectors

(6.46) $\quad \mathbf{abc} = \mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c}$

From the origin 2-vector product (5.59) of two 1-vectors $\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}$, we get

(6.47) $\quad \mathbf{a}(\mathbf{b}\cdot\mathbf{c} + \mathbf{b}\wedge\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b})\mathbf{c} = \mathbf{abc}$

and by applying (6.40) with the distributive rules (5.39) and (5.40) we have

(6.48) $$\underbrace{\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})}_{\langle\mathbf{abc}\rangle_1} + \underbrace{\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})}_{\langle\mathbf{abc}\rangle_3} = \underbrace{(\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c}}_{pqg\text{-}1} + \underbrace{(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}}_{pqg\text{-}3} = \underset{\textit{qualities of directions.}}{\mathbf{abc} \in \mathcal{G}_3^-(\mathbb{R})}.$$

For this simple product **abc**, the scalar $\langle\mathbf{abc}\rangle_0 = 0$ and the bivector $\langle\mathbf{abc}\rangle_2 = 0$ part vanish.
At orthogonality $\langle\mathbf{abc}\rangle_1 = 0$, then we just have $\mathbf{abc} = \langle\mathbf{abc}\rangle_3 = \pm\,|\mathbf{abc}|\boldsymbol{i}$ with chiral ***direction***.

6.2.5.5. Even and Odd Multivector in general

For the 𝔓 plane concept, we from (5.162) have resolved a general 2-multivector as

(6.49) $\quad A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 \quad \in \mathcal{G}_2(\mathbb{R}).$

For the 𝔷-space concept, we resolve a general 3-multivector as

(6.50) $\quad A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \langle A\rangle_3 \quad \in \mathcal{G}_3(\mathbb{R}).$

A generalised a n-multivector in $\mathcal{G}_n(\mathbb{R})$ we resolve in ***grades*** r and write it as

(6.51) $\quad A = \sum_r^n \langle A\rangle_r = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \langle A\rangle_3 + \cdots + \langle A\rangle_n \quad \in \mathcal{G}_n(\mathbb{R}).$

A multivector that is simply *graded* as $A = \langle A\rangle_r = A_{\bar{r}}$ is called homogeneous of ***grade*** r and often named a r-blade or just a simple r-vector with a ***primary quality of*** r ***'th grade***.
A r-blade or a simple r-vector representing a ***pqg-***r ***direction***.
E.g., simply: trivector as 3-blade, bivector as a 2-blade, 1-vector as a 1-blade, and scalar as 0-blade
Later we will introduce a 4-vector as a 4-blade, etc.
The multivector A is called

- *odd* when $\langle A\rangle_r = 0$ for all even r, $\langle A\rangle_- = \langle A\rangle_1 + \langle A\rangle_3 + \cdots$ or
- *even* when $\langle A\rangle_r = 0$ for all odd r, $\langle A\rangle_+ = \langle A\rangle_0 + \langle A\rangle_2 + \cdots$

In general, all multivectors can be separated

(6.52) $\quad A = \langle A\rangle_+ + \langle A\rangle_-$

Now we see that (5.59) $\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}$ is even, $\mathbf{ab} = \langle\mathbf{ab}\rangle_+ = \langle\mathbf{ab}\rangle_0 + \langle\mathbf{ab}\rangle_2$,
and that (6.46)-(6.48) is odd, $\mathbf{abc} = \langle\mathbf{abc}\rangle_-$, as well as the 1-vector $\mathbf{a} = \langle\mathbf{a}\rangle_-$ is odd.

6.2.5.6. Product of Two Bivectors

We look at two bivectors $\mathbf{B}$ and $\mathbf{A} = \mathbf{a}_2\wedge\mathbf{a}_1 = \mathbf{a}_2\mathbf{a}_1$ (where $\mathbf{a}_2\cdot\mathbf{a}_1 = 0$). We form the product

(6.53) $$\begin{aligned}\mathbf{AB} &= \mathbf{a}_2\mathbf{a}_1\mathbf{B} = \mathbf{a}_2(\mathbf{a}_1\cdot\mathbf{B} + \mathbf{a}_1\wedge\mathbf{B}) = \mathbf{a}_2\cdot(\mathbf{a}_1\cdot\mathbf{B}) + \mathbf{a}_2\wedge(\mathbf{a}_1\cdot\mathbf{B}) + \mathbf{a}_2\cdot(\mathbf{a}_1\wedge\mathbf{B}) + \mathbf{a}_2\wedge\mathbf{a}_1\wedge\mathbf{B}\\ &= \mathbf{A}\cdot\mathbf{B} + \tfrac{1}{2}(\mathbf{a}_2\mathbf{a}_1\mathbf{B} - \mathbf{B}\mathbf{a}_2\mathbf{a}_1) + \mathbf{A}\wedge\mathbf{B} = \mathbf{A}\cdot\mathbf{B} + \tfrac{1}{2}(\mathbf{AB} - \mathbf{BA}) + \mathbf{A}\wedge\mathbf{B} = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\times\mathbf{B} + \mathbf{A}\wedge\mathbf{B}.\end{aligned}$$

Where we have used (6.42) and introduced the commutator product

(6.54) $\quad \mathbf{A}\times\mathbf{B} = \tfrac{1}{2}(\mathbf{AB} - \mathbf{BA}).$

This deduction works for any simple ***grade*** r-blade $B = \langle B\rangle_r = B_{\bar{r}}$, special $\mathbf{B} = \langle\mathbf{B}\rangle_2$, [13]p.10,(1.37).
The inner product of bivectors is a scalar $\mathbf{A}\cdot\mathbf{B} = \langle\mathbf{AB}\rangle_0 \in \mathbb{R}$, just like two equal grade-r-blades.[296]
In a pure 𝔷-space due to (6.35) the outer product of two bivectors vanish $\mathbf{A}\wedge\mathbf{B} = 0$.

[296] This is taken from [13]p.6,(1.21) for the general definition of the inner product of blades $A_{\bar{r}}\cdot B_{\bar{s}} \equiv \langle A_{\bar{r}}B_{\bar{s}}\rangle_{|r-s|}$, if $r,s > 0$.

6.2.5.7. Commutator Product of Multivectors in Geometric Algebra

The commutator product of two multivectors A and B we in geometric algebra define as

(6.55) $\quad A\times B = \tfrac{1}{2}(AB - BA)$

For further details for higher ***grades*** please consult the literature, e.g., [10], [13], [18].
Here we mention the Jacobi identity

(6.56) $\quad A\times(B\times C) + B\times(C\times A) + C\times(A\times B) = 0$

Special for a bivector B and a general 1-vector a we as (6.42) have the commutation

(6.57) $\quad B\times a = \tfrac{1}{2}(Ba - aB) = B\cdot a = -a\cdot B = -\tfrac{1}{2}(aB - Ba) = -a\times B$

When we later use this (6.55) commutating product in a quantum mechanical context we use the nomenclature

(6.58) $\quad [A\times B] = \tfrac{1}{2}(AB - BA) \quad$ or $\quad 2[A\times B] = AB - BA$

for multivector commutator relations to keep the correspondence with the quantum mechanical commutator relation we first defined in chapter I. (2.50) $[\mathrm{b},\mathrm{d}] = \mathrm{b}\,\mathrm{d} - \mathrm{d}\,\mathrm{b}.$

It is worth noting the difference by the factor of one-half and thinking, what impact does this have on physics?