For a commutation of the three 1-vector outer product sequences in a trivector we have

(6.17) $$\mathsf{c}\wedge\mathsf{b}\wedge\mathsf{a} = \mathsf{a}\wedge\mathsf{c}\wedge\mathsf{b} = \mathsf{b}\wedge\mathsf{a}\wedge\mathsf{c} = -\mathsf{c}\wedge\mathsf{a}\wedge\mathsf{b} = -\mathsf{b}\wedge\mathsf{c}\wedge\mathsf{a} = -\mathsf{a}\wedge\mathsf{b}\wedge\mathsf{c} = \mathsf{T}.$$

The two opposite sequential cyclical permutation orientations correspond to the two opposite chiral orientations $\pm$ of space structure volume ***direction*** $\pm\mathsf{T}$.Figure 6.7

6.2.2.2. The Magnitude of a Trivector

The trivector $\mathsf{T} = \mathsf{c}\wedge(\mathsf{b}\wedge\mathsf{a}) = \mathsf{c}\wedge\mathsf{B}$ is the outer product of a 1-vector c to a bivector $\mathsf{B} = (\mathsf{b}\wedge\mathsf{a})$.
The square auto product of this bivector is as (5.66) $\mathsf{B}^2 = -|\mathsf{B}|^2 = -|\mathsf{b}|^2|\mathsf{a}|^2\sin^2\theta \quad \leq 0$, where the arc in the intuited bivector object is $\theta = \measuredangle(\mathsf{b}\wedge\mathsf{a})$, which parallelogram area $|\mathsf{B}| = |\mathsf{b}\wedge\mathsf{a}| = |\mathsf{b}||\mathsf{a}||\sin\theta| \quad \geq 0 \in \mathbb{R}$, then represent the bivector magnitude.
We intuit the third 1-vector c in the angle arc $\varphi = \measuredangle(\mathsf{c}\wedge\mathsf{B})$ in the sense of E XI.De.5. and get the height of the prism $|\mathsf{c}_{\perp(\mathsf{b}\wedge\mathsf{a})}| = |\mathsf{c}||\sin\varphi| \quad \geq 0 \in \mathbb{R}$, with the volume

(6.18) $$|\mathsf{T}| = |\mathsf{c}\wedge\mathsf{b}\wedge\mathsf{a}| = |\mathsf{c}||\mathsf{b}||\mathsf{a}||\sin\theta||\sin\varphi| \quad \geq 0 \in \mathbb{R}$$

This volume object for our intuition represents the magnitude of the trivector idea.

6.2.3. The Trivector and the $\mathfrak{Z}$-space Chiral Pseudoscalar

From this, we define the trivector volume square

(6.19) $$\mathsf{T}^2 = \mathsf{T}\cdot\mathsf{T} = |\mathsf{c}_{\perp(\mathsf{b}\wedge\mathsf{a})}|^2\mathsf{B}^2 = -|\mathsf{B}|^2|\mathsf{c}|^2|\sin\varphi|^2 = -|\mathsf{c}|^2|\mathsf{b}|^2|\mathsf{a}|^2\sin^2\theta\sin^2\varphi \quad \leq 0.$$

The conclusion is $\mathsf{T}^2 = -|\mathsf{T}|^2 \leq 0$, and we call the trivector a *chiral pseudoscalar* for the $\mathfrak{Z}$-space.
This simple trivector $\mathsf{T} = \mathsf{c}\wedge\mathsf{b}\wedge\mathsf{a}$ is also often called a 3-blade. The external product represents a ***primary quality of third grade*** (***pqg***-3) that gives the $\mathfrak{Z}$-space ***chiral direction*** of a volume.
The orientation of this ***pqg***-3 ***direction*** volume can be reversed as expressed in (6.15)-(6.17) by the two states $\mathsf{T} = \pm|\mathsf{T}|$.
This is the fundamental principle of the permutation orientation of three 1-vector ***directions***.

6.2.3.2. The Cartesian Orthonormal Basis 1-vector Set of *Primary Quality of Third Grade* (*pqg*-3)

A three-dimensional Euclidean geometric 1-vector space $\mathcal{E}_3 \rightarrow (V_3, \mathbb{R})$ can be generated from a dextral orthonormal set of geometric basis $\{\sigma_1, \sigma_2, \sigma_3\}$ for a Cartesian system where $\sigma_2 \perp \sigma_1$, $\sigma_3 \perp \sigma_2$, $\sigma_1 \perp \sigma_3$ are perpendicular and $|\sigma_1| = |\sigma_2| = |\sigma_3| = 1$.
By the geometric vector product algebra, this is expressed as

(6.20) $$\sigma_j\cdot\sigma_k = \frac{1}{2}(\sigma_j\sigma_k + \sigma_k\sigma_j) = \delta_{jk} \qquad \text{where} \quad \delta_{jk} = \begin{cases} 1 \text{ for } j = k \\ 0 \text{ for } j \neq k \end{cases}$$

From this Euclidean $\mathcal{E}_3$ orthonormal basis of three 1-vectors we form a *unit dextral chiral pseudoscalar trivector* for the $\mathfrak{Z}$-space

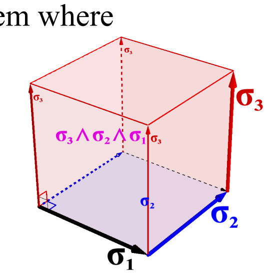

Figure 6.8 The unit trivector ***direction*** formed on the dextral orthonormal basis $\{\sigma_1, \sigma_2, \sigma_3\}$ as an object in $\mathfrak{Z}$ space.

(6.21) $$\boldsymbol{i} = \sigma_3\wedge\sigma_2\wedge\sigma_1 \quad = \quad \boldsymbol{i} \coloneqq \sigma_3\sigma_2\sigma_1$$

This chirality as the outer product $\sigma_3\wedge\sigma_2\wedge\sigma_1$ shown in Figure 6.8 is an external view of a dextral object basis for us.
Further the orthonormality (6.20) for this dextral object basis gives the scalar products in this to zero 0, hence we archive this simple operator product for the *chiral volume unit pseudoscalar*

(6.22) $$\boxed{\boldsymbol{i} \coloneqq \sigma_3\sigma_2\sigma_1}.$$

This is a simple left-order sequential operator product: First we operate with σ_1, then to the left multiplication with the operator σ_2 and further left multiplying with the operator σ_3.
This is equal to the outer dextral chiral product $\sigma_3\wedge\sigma_2\wedge\sigma_1$.
In all, we have the unit chiral pseudoscalar volume operator $\boldsymbol{i} \coloneqq \sigma_3\sigma_2\sigma_1 = \sigma_3\sigma_2\sigma_1$ in Figure 6.9, an unit *amoeba*.

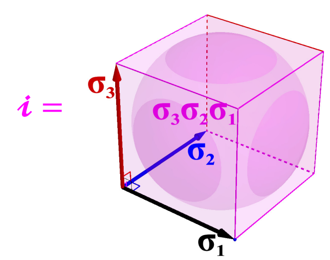

Figure 6.9 The unit chiral pseudoscalar subject as the ***pqg***-3 ***direction quality*** of the $\mathfrak{Z}$ space substance: $\boldsymbol{i} \equiv \sigma_3\sigma_2\sigma_1$.
Any *amoeba* $|\boldsymbol{i}| \equiv 1$ is a representative.

This unit pseudoscalar represents a ***chirality direction*** as the ***primary quality of third grade*** (***pqg***-3) of the $\mathfrak{Z}$-space with a dextral (righthanded) orientation, that is expressed through the left-operated sequential order 1,2,3 of the algebraic multiplication operations of the 1-vectors σ_1, σ_2 and then σ_3.
The reversed sinistral (lefthanded) orientation of the volume ***pqg***-3 ***direction*** we express by the reversed commutated order

(6.23) $$\boxed{-\boldsymbol{i} \coloneqq \sigma_1\sigma_2\sigma_3} \quad = \quad \widetilde{\sigma_3\sigma_2\sigma_1}.$$

where the left operated sequential order is 3,2,1.[292]
The idea of this trivector chiral pseudoscalar concept for our intuition always a priori depends on the orthonormal basis $\{\sigma_1, \sigma_2, \sigma_3\}$ object for the $\mathfrak{Z}$-space substance ***direction*** considered in our minds. See this dextral orientated sequential object Figure 6.9, (right-hand rule: 1-thumb, 2-index, 3-long)
Although this ***chiral-directed*** unit volume by definition is linked to the unit cube as an object, orientated by (6.22) or (6.23), the unit chiral pseudoscalar subject $\boldsymbol{i}$ can take any *amoeba* $\boldsymbol{i}$ form in the $\mathfrak{Z}$-space substance, as long as its volume is one unit $|\boldsymbol{i}| \coloneqq 1$, and orientation is preserved.

To simplify intuition, we must demand the dextral object basis orthogonal:
$\sigma_1\cdot\sigma_2 = \sigma_2\cdot\sigma_3 = \sigma_3\cdot\sigma_1 = 0$, and normalisation: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \Rightarrow |\sigma_1| = |\sigma_2| = |\sigma_2| = 1$.

The chiral unit pseudoscalar characteristic is given by the fact $\boldsymbol{i}^2 = -1$ because

(6.24) $$\boldsymbol{i}\boldsymbol{i} = \boldsymbol{i}^2 = (\sigma_3\sigma_2\sigma_1)^2 = \sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 = \sigma_3\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\sigma_1 = -\sigma_2\sigma_2\sigma_1\sigma_1 = -1,$$

or alternative expressed

(6.25) $$\boldsymbol{i}^2 = (\sigma_3\sigma_2\sigma_1)^2 = -(\sigma_1\sigma_2\sigma_3)(\sigma_3\sigma_2\sigma_1) = -\sigma_1\sigma_2\sigma_3\sigma_3\sigma_2\sigma_1 = -\sigma_1\sigma_2\sigma_2\sigma_1 = -\sigma_1\sigma_1 = -1.$$

This leads to the normalized ***pqg***-3 volume magnitude $|\boldsymbol{i}| = |-\boldsymbol{i}| = 1$.
We remember from (5.72) the plane unit anticommuting pseudoscalar

(6.26) $$i = \sigma_2\wedge\sigma_1 = \frac{1}{2}(\sigma_2\sigma_1 - \sigma_2\sigma_1) = \sigma_2\sigma_1$$

this is the unit bivector for the plane ***pqg***-2 ***direction*** $\widehat{\mathsf{B}} = i = \sigma_2\sigma_1$. The unit trivector chiral pseudoscalar can then be written $\boldsymbol{i} = \sigma_3 i$. Geometric interpreted $\sigma_3 \perp i$, where σ_3 is normal to i, that is the transversal unit bivector to the normal 1-vector σ_3.
We remember from § 5.2.6.4 that the form of the plane subject i has any arbitrarily shaped plane amoeba as object i as long as this has unit magnitude $|i| = 1$, illustrated in Figure 5.14. The same will be the case for any shaped *solid volume amoeba* subject $\boldsymbol{i}$ with magnitude $|\boldsymbol{i}| = 1$ that will represent the idea of the ***pqg***-3 ***direction*** *subject* $\boldsymbol{i}$ of the $\mathfrak{Z}$-space *substance*. We have both chiral orientations $\boldsymbol{i}$ and $-\boldsymbol{i}$ which are the two stats of the unitary trivector idea.
The chiral ***direction*** of a unit space solid volume $\widehat{\mathsf{T}}$ has two eigenstates

(6.27) $$\widehat{\mathsf{T}} = \pm\boldsymbol{i} = \pm 1\boldsymbol{i}, \qquad \text{in that} \quad \widehat{\mathsf{T}}^2 = \boldsymbol{i}^2 = -|\boldsymbol{i}|^2 = -|\widehat{\mathsf{T}}|^2 = -1$$

We say that the unit-space-solid-volume ***direction*** $\widehat{\mathsf{T}}$ has two eigenvalues 1 and -1.
Comparing with quantum mechanics we intuit $\boldsymbol{i}$ as a ***direction*** *operator* for a unit-space-solid with *two chiral orientation eigenstates*.
Any arbitrary space volume $\upsilon = |\mathsf{T}| \geq 0$ provided by a trivector $\mathsf{T} = \upsilon\widehat{\mathsf{T}}$ for a space-solid ***pqg***-3 ***direction*** thus has two eigenstates $\mathsf{T}^+ = +\upsilon\boldsymbol{i}$ or $\mathsf{T}^- = -\upsilon\boldsymbol{i}$ and the ***quantitative*** eigenvalues $+\upsilon$ and $-\upsilon$ for the volume. When you have a solid volume, you should seriously consider its orientation and which of the two trivectors T or $-\mathsf{T}$ you use in the chiral intuition.
The operator $\boldsymbol{i}$ acts on the space concept $\mathfrak{G}$ and creates one $\mathfrak{Z}$-space ***direction***.

[292] In this book, we use left multiplication of the sequential operational order. This is the reversed order of that first defined by David Hestenes [6](6.3)p.16, [10](3.2), etc. $i = \sigma_1\wedge\sigma_2\wedge\sigma_3$. The standard of this book is therefore $\boldsymbol{i} = -i$, opposite the Hestenes tradition.

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