These four ***outwards*** 1-vector ***directions*** from an arbitrary locus situs point O, we call a **tetraon**. The tetraon points out the four vertexes of a tetrahedron and by that the circumscribed sphere. Is the tetrahedron regular symmetric the 1-vectors in the tetraon fulfil $\mathbf{u}_a+\mathbf{u}_b+\mathbf{u}_c+\mathbf{u}_d=0 \Rightarrow$

(6.9) $\mathbf{u}_a=-(\mathbf{u}_b+\mathbf{u}_c+\mathbf{u}_d),\quad \mathbf{u}_b=-(\mathbf{u}_c+\mathbf{u}_d+\mathbf{u}_a),\quad \mathbf{u}_c=-(\mathbf{u}_d+\mathbf{u}_a+\mathbf{u}_b),\quad \mathbf{u}_d=-(\mathbf{u}_a+\mathbf{u}_b+\mathbf{u}_c).$

This symmetry is well-known for the four valent carbon atom in a methane molecule. In general, given a locus situs center O and three arbitrary linear independent ***pqg***-1 ***directions*** given by three unit 1-vectors $\mathbf{u}_a,\mathbf{u}_b,\mathbf{u}_c$, then a fourth ***pqg***-1 ***direction*** in $\mathfrak{Z}$-space can **be** spanned from these

(6.10) $\mathbf{r}=\ \alpha^a\mathbf{u}_a+\alpha^b\mathbf{u}_b+\alpha^c\mathbf{u}_c$ for by *contravariant coordinates* $\forall\alpha^a,\alpha^b,\alpha^c\in\mathbb{R}$.[289]

For non-orthogonality, we note $r=|\mathbf{r}|\neq\sqrt{\alpha^{a2}+\alpha^{b2}+\alpha^{c2}}$. Anyway for $\lambda^a=\frac{\alpha^a}{r},\ \lambda^b=\frac{\alpha^b}{r}$, and $\lambda^c=\frac{\alpha^c}{r}$, from these, we form unit radius-1-vector $\mathbf{u}=\mathbf{r}/r=\lambda^a\mathbf{u}_a+\lambda^b\mathbf{u}_b+\lambda^c\mathbf{u}_c$, that from all possibilities spans a unit sphere, so that the fourth ***direction*** from a center point out by these $\mathbf{r}=r\mathbf{u}$ in $\mathfrak{Z}$-space. In this S^2 spherical symmetric[290] in space, the fourth 1-vector is linearly dependent on the other three 1-vectors as a basis $\{\mathbf{u}_a,\mathbf{u}_b,\mathbf{u}_c\}$.

To imagine this $\mathfrak{Z}$ symmetry, the reader can refer to the plane object in Figure 5.32 and Figure 5.33 and extrapolate the fourth ***direction*** out of the figure plane.

6.1.4.2. The Six Bivector Angular Planes of the Regular Tetraon

Like in (5.115) we will look at the 1-rotor planes made by the mutual pair products of the four 1-vector ***directions*** $\mathbf{u}_a,\mathbf{u}_b,\mathbf{u}_c,\mathbf{u}_d$, each consisting of two 1-vectors representing the 1-rotors, which we can split into scalars and bivector

(6.11)
$$\begin{aligned}
\mathbf{u}_c\mathbf{u}_b &= \mathbf{u}_c\cdot\mathbf{u}_b+\mathbf{u}_c\wedge\mathbf{u}_b\,, & \mathbf{u}_a\mathbf{u}_b &= \mathbf{u}_a\cdot\mathbf{u}_b+\mathbf{u}_a\wedge\mathbf{u}_b\,,\\
\mathbf{u}_d\mathbf{u}_c &= \mathbf{u}_d\cdot\mathbf{u}_c+\mathbf{u}_d\wedge\mathbf{u}_c\,, & \mathbf{u}_a\mathbf{u}_c &= \mathbf{u}_a\cdot\mathbf{u}_c+\mathbf{u}_a\wedge\mathbf{u}_c\,,\\
\mathbf{u}_b\mathbf{u}_d &= \mathbf{u}_b\cdot\mathbf{u}_d+\mathbf{u}_b\wedge\mathbf{u}_d\,, & \mathbf{u}_a\mathbf{u}_d &= \mathbf{u}_a\cdot\mathbf{u}_d+\mathbf{u}_a\wedge\mathbf{u}_d\,.
\end{aligned}$$

For the *regular* central symmetric *tetraon* Figure 6.5 where we demand $\mathbf{u}_a+\mathbf{u}_b+\mathbf{u}_c+\mathbf{u}_d=0$ (6.9), and achieve equal mutual angles β (~109.5°) with the 1-rotor split

(6.12)
$$\begin{aligned}
\mathbf{u}_c\mathbf{u}_b &= -\tfrac{1}{3}+\mathbf{u}_c\wedge\mathbf{u}_b\,,\ \lozenge, & \mathbf{u}_a\mathbf{u}_b &= -\tfrac{1}{3}+\mathbf{u}_a\wedge\mathbf{u}_b\,,\ \lozenge,\\
\mathbf{u}_d\mathbf{u}_c &= -\tfrac{1}{3}+\mathbf{u}_d\wedge\mathbf{u}_c\,,\ \blacklozenge, & \mathbf{u}_a\mathbf{u}_c &= -\tfrac{1}{3}+\mathbf{u}_a\wedge\mathbf{u}_c\,,\ \lozenge,\\
\mathbf{u}_b\mathbf{u}_d &= -\tfrac{1}{3}+\mathbf{u}_b\wedge\mathbf{u}_d\,,\ \blacklozenge, & \mathbf{u}_a\mathbf{u}_d &= -\tfrac{1}{3}+\mathbf{u}_a\wedge\mathbf{u}_d\,,\ \lozenge.
\end{aligned}$$

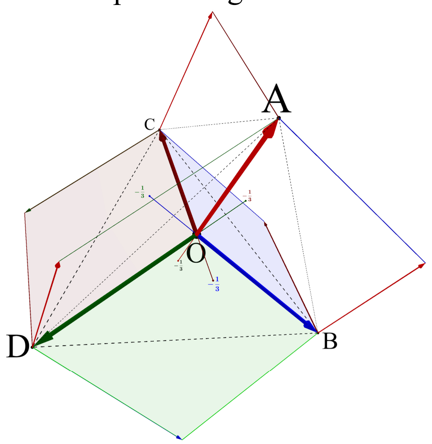

Figure 6.5 Regular tetraon demand by 1-rotor split in scalar and bivector (6.12).

The scalar number $-\frac{1}{3}=\cos\beta$ given from the four mutual angles between the 1-vectors are mutual *covariant* coordinates for the basis set $\{\mathbf{u}_a,\mathbf{u}_b,\mathbf{u}_c,\mathbf{u}_d\}$ itself. These are the normal *distances* from O to the faces of the endowed regular tetrahedron. Faces, that are *transversal planes*[291] to this 1-vector basis.

We use the projection operator (5.184) $P_{\mathbf{a}}\mathbf{x}=(\mathbf{x}\cdot\mathbf{a})\mathbf{a}^{-1}\rightarrow$ e.g. $P_{\mathbf{u}_a}\mathbf{u}_b=(\mathbf{u}_b\cdot\mathbf{u}_a)\mathbf{u}_a^{-1}=-\frac{1}{3}\mathbf{u}_a^{-1}$.

Multiplying this by $\mathbf{u}_a$ just gives the mutual *covariant* coordinate $\mathbf{u}_aP_{\mathbf{u}_a}\mathbf{u}_b=-\frac{1}{3}$ for this basis.

By normalising this regular tetraon basis $\mathbf{u}_a^2=\mathbf{u}_b^2=\mathbf{u}_c^2=\mathbf{u}_d^2=1$, we get that e.g. $\mathbf{u}_a=\mathbf{u}_a^{-1}$.

The *covariant* sum in ***direction*** $\mathbf{u}_a$ simply is $P_{\mathbf{u}_a}\mathbf{u}_a+P_{\mathbf{u}_a}\mathbf{u}_b+P_{\mathbf{u}_a}\mathbf{u}_c+P_{\mathbf{u}_a}\mathbf{u}_d=+1-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}=0$

The *contravariant* sum in ***direction*** $\mathbf{u}_a$ is $1\mathbf{u}_a+0\mathbf{u}_b+0\mathbf{u}_c+0\mathbf{u}_d=\mathbf{u}_a$, and the full *contravariant* sum for the regular tetraon basis is $1\mathbf{u}_a+1\mathbf{u}_b+1\mathbf{u}_c+1\mathbf{u}_d=0$ just as the demand (6.9).

Only three of the six bivector planes defined by the 1-rotors (6.11) are necessary to give a unique intersection definition of a locus center origo O.

– Below we set $\mathbf{u}_x\cdot\mathbf{u}_y=0$, this gives a Cartesian basis.

[289] Upper indices are used to emphasise contravariant coordinates, whereas lower indices indicate covariant coordinates.
[290] The name S^2 for the spherical symmetric has its origin in two angular *spherical coordinates* $(1,\theta,\phi)$ for a unit sphere.
[291] We will gradually realise that such an idea of ***transversal plane directions*** is a very fundamental concept for physics!

6.2. The Geometric Algebra of Natural Space

In the tradition natural space has been represented by the 1-vector space $(V_3,\mathbb{R})$, $\dim(V_3)=3$ of 3-dimensions for any extension *length*, *breadth*, and *depth*.[158] We demand the natural 3-dimensional space V_3 of physics as Euclidean $\mathcal{E}_3$, where the auto product $\mathbf{v}^2=\mathbf{v}\cdot\mathbf{v}\geq 0$ for all 1-vectors $\forall\mathbf{v}\in V_3$ are positive definite, setting the metric signature $\epsilon_A=+1$, referring to § 5.2.1.5. For $\mathfrak{Z}$-space we will expand this 3-dimensional view of 1-vectors with a linear geometric algebra of higher dimensions as we did for the pure plane concept with the scalar concept and the bivector concept that extensively in the plane idea was imagined as an anticommuting pseudoscalar concept.

6.2.1. Addition of Bivectors

In $\mathfrak{Z}$-space we accept bivectors from several independent planes. We take start with three linear independent 1-vectors **a**, **b**, **c** and make two linear independent bivectors out of these **b**∧**a** and **c**∧**a**. The addition of these bivectors is defined by the distributive rule

(6.13) $\mathbf{b}\wedge\mathbf{a}+\mathbf{c}\wedge\mathbf{a}\ =(\mathbf{c}+\mathbf{b})\wedge\mathbf{a}$

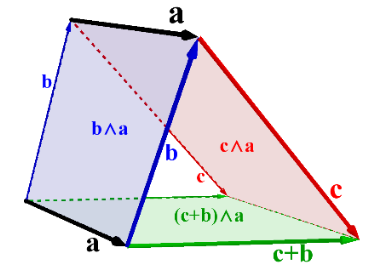

Figure 6.6 Bivector addition from the foundation on the external product definition of 1-vectors, that comply with the distributive law.

This is in Figure 6.6 shown as the object $(\mathbf{c}+\mathbf{b})\wedge\mathbf{a}$, where the sum of two bivectors is again a bivector.

In this 3-dimensional 1-vector structure, the interpretation of the bivector $\mathbf{B}=\beta\boldsymbol{i}$ as a plane pseudoscalar loos its specific meaning.

The two independent plane ***directions*** given by the two unit bivectors $\boldsymbol{i}_{ba}=\widehat{\mathbf{b}\wedge\mathbf{a}}$ and $\boldsymbol{i}_{ca}=\widehat{\mathbf{c}\wedge\mathbf{a}}$ can by linear combination give any plane ***direction*** concerning the common 1-vector ***direction*** **a**,

(6.14) $\mathbf{X}_a=\beta\boldsymbol{i}_{ba}+\alpha\boldsymbol{i}_{ca}$

Anny two plane ***pqg***-2 ***directions*** form together an intersection ***pqg***-1-vector ***direction***.

That can be intuited by their 1-vector objects as in Figure 6.6, compare § 6.1.2,t. and Figure 6.2, and is given from E XI.De.6. and especially E XI.Pr.3.

We see the translation of the object **a** represents the subject ***pqg***-1-vector ***direction*** **a**.

The same for **b** and **c**, and further for the bivector subjects **b**∧**a**, **c**∧**a**, and (**c**+**b**)∧**a**, we will intuit as translation invariant objects concerned in their respective supported subject planes.

Anyway, as a foundation, we shall take start in the 3-dimensional set of three linear independent geometric 1-vector ***directions*** as a basis set for a Euclid vector space $(V_3,\mathbb{R})\leftrightarrow\mathcal{E}_3$, representing a classical local extension (e.g., as a Descartes system).

6.2.2. The Trivector concept

From three linear independent geometric 1-vector objects **a**, **b**, **c** we form a solid prism as shown in Figure 6.7.

First, we let 1-vector **b** operate on 1-vector **a** to form the bivector **b**∧**a** . Then we let the 1-vector **c** operate on this bivector and get a trivector $\mathbf{T}=\mathbf{c}\wedge(\mathbf{b}\wedge\mathbf{a})$ representing an oriented volume spanned by these three 1-vectors.

In Figure 6.7 this solid volume object is marked **c**∧**b**∧**a**.

This outer multiplication shall obey the associative rule

(6.15) $\mathbf{c}\wedge(\mathbf{b}\wedge\mathbf{a})=(\mathbf{c}\wedge\mathbf{b})\wedge\mathbf{a}\ \ =\mathbf{T}\,.$

Therefore, we often just use the nomenclature **c**∧**b**∧**a** for a trivector made from the three 1-vectors left sequence.

By commutation of the 1-vectors,l we from (5.58) have

(6.16) $\mathbf{c}\wedge\mathbf{a}\wedge\mathbf{b}=\ \mathbf{c}\wedge(\mathbf{a}\wedge\mathbf{b})=\mathbf{c}\wedge(-\mathbf{b}\wedge\mathbf{a})\ \ =-\mathbf{T}\,.$

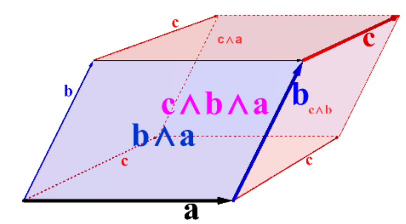

Figure 6.7 A trivector object **c**∧**b**∧**a** is formed and spanned by the three 1-vector objects **a**, **b**, **c**.
This prism is an example of a more general formless but ***directional*** trivector volume subject in the substance idea of the $\mathfrak{Z}$ space concept. – OBS: This displayed geometric object is sinistral, so that $\mathbf{T}=\mathbf{c}\wedge\mathbf{b}\wedge\mathbf{a}$ has a negative chiral orientation.