5.8. The Exponential Function of Arbitrary Multivectors

The generalised multivector A can be dissolved in its finite ***grade*** components $\langle A\rangle_r$

(5.387) $$A=\sum_{r=0}^{n}\langle A\rangle_r=\langle A\rangle_0+\langle A\rangle_1+\langle A\rangle_2+\cdots+\langle A\rangle_n \quad \in \mathcal{G}_n=\mathcal{G}_n(\mathbb{R}).$$

Note that each $\langle A\rangle_r$ of ***grade-r*** for $2<r\le n$ is an external extension to the plane concept.
We now define a generalisation of the exponential function for multivectors by a power series

(5.388) $$\exp(A) \;=\; e^{A} \;=\sum_{k=0}^{\infty}\frac{A^k}{k!}=1+\frac{A}{1!}+\frac{A^2}{2!}+\frac{A^3}{3!}+\cdots+\frac{A^k}{k!}+\cdots$$

This is an algebraic definition of the exponential function and if the multivector $A\in\mathcal{G}_n$ stays in the algebra $\mathcal{G}_n$ the result $Z=e^A\in\mathcal{G}_n$ stays inside the same algebra. The reason is that for all ***grades*** higher than n disappears $\langle A\rangle_{>n}=0$, and the full geometric algebra $\mathcal{G}_n$ is closed.
The same closed consistency for the even algebras $\mathcal{G}_n^+$, with $A^+=\langle A\rangle_0+\langle A\rangle_2+\cdots$.

For two different multivectors $A,B\in\mathcal{G}_n$ or $\mathcal{G}_n^+$, we formulate the *restrictive* additive rule for the exponentials

(5.389) $$e^Ae^B=e^{(A+B)}=e^{(B+A)}=e^Be^A$$

The *restriction* is that products commute $AB=BA$, just as addition always does $A+B=B+A$.
This is seen by the binomial series for the added multivectors with the integer exponents

(5.390) $$(B+A)^n=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}A^{n-k}B^k,\quad \text{with the demand } AB=BA,\ \text{as [10]p.73 resulting in}$$

(5.391) $$e^Ae^B=\sum_{m=0}^{\infty}\frac{A^m}{m!}\cdot\sum_{n=0}^{\infty}\frac{B^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{A^{n-k}}{(n-k)!}\frac{B^k}{k!}=\sum_{n=0}^{\infty}\frac{(A+B)^n}{n!}=e^{(A+B)}=e^{(B+A)}.$$

For the non-commutating multivectors $AB\neq BA$, we will always be able to write

(5.392) $$e^Ae^B=e^C,$$

but then we have $C\neq A+B$, and we do not have a general solution. Later in this book we will study the case with *anticommuting* $AB=-BA$ bivector exponentials for rotations in an even algebra $\mathcal{G}_3^+\sim\mathcal{G}_{0,2}$ containing several independent plane ***directions***.

5.8.1.1. The Hyperbolic Functions of Multivectors

We separate the exponent power series (5.388) in even and odd powers by the definitions

(5.393) $$\cosh A := \frac{e^A+e^{-A}}{2}=\sum_{k=0}^{\infty}\frac{A^{2k}}{(2k)!}=1+\frac{A^2}{2!}+\frac{A^4}{4!}+\frac{A^6}{6!}+\cdots$$

(5.394) $$\sinh A := \frac{e^A-e^{-A}}{2}=\sum_{k=0}^{\infty}\frac{A^{2k+1}}{(2k+1)!}=A+\frac{A^3}{3!}+\frac{A^5}{5!}+\frac{A^7}{7!}+\cdots$$

The sum of the even and the odd hyperbolic part is the exponential series (5.388)

(5.395) $$e^A=\cosh A+\sinh A.$$

The input multivector A is the argument in the functions $e^A=\exp(A)$, $\cosh(A)$ and $\sinh(A)$

5.8.1.2. The *cosine* and *sine* Functions of Arbitrary $\mathcal{G}_n$ Multivectors

Introducing the generalised multivector unit pseudoscalar I with the founding property $I^2=-1$.
Take it and (left) operate by multiplying it to a multivector A we get a new multivector $B=IA$.
We will use this to transform (5.393)-(5.394) to the general *cosine* and *sine* multivector functions

(5.396) $$\cos A=\cosh IA=1+\frac{(IA)^2}{2!}+\frac{(IA)^4}{4!}+\frac{(IA)^6}{6!}+\cdots=\sum_{k=0}^{\infty}\frac{(-1)^kA^{2k}}{(2k)!}=1-\frac{A^2}{2!}+\frac{A^4}{4!}-\frac{A^6}{6!}+\cdots.$$

(5.397) $$\sin A=\frac{1}{I}\sinh IA=A+\frac{(IA)^3}{3!\,I}+\frac{(IA)^5}{5!\,I}+\frac{(IA)^7}{7!\,I}+\cdots=\sum_{k=0}^{\infty}\frac{(-1)^kA^{2k+1}}{(2k+1)!}=1-\frac{A^3}{3!}+\frac{A^5}{5!}-\frac{A^7}{7!}+\cdots.$$

We compare with the traditional definition formulas, that also is valid for multivectors

(5.398) $$\cos A=\frac{e^{IA}+e^{-IA}}{2}\quad\text{and}\quad \sin A=\frac{e^{IA}+e^{-IA}}{2I},$$

and back to the exponential function with the multivector argument $B=IA$

(5.399) $$e^{IA}=\cos A+I\sin A.$$

Here it is essential to distinguish the different pseudoscalar units $I=\langle I\rangle_n$ all with the same ***quality*** $I^2=-1$ of the different ***direction primary quality*** for each ***maximal grade*** n for the geometric algebras $\mathcal{G}_n$ of physics. For the Euclidian plane ***direction*** unit $\boldsymbol{i}$ we have $\boldsymbol{i}^2=-1$.
More about these pseudoscalars for higher maximal ***grades*** later below chapters 6, 7, etc…

5.8.2. Exponential and Hyperbolic Functions in the Plane Concept

In the plane concept, the multivector A can be dissolved in its ***grade*** components $\langle A\rangle_r$

(5.400) $$A=\sum_{r=0}^{2}\langle A\rangle_r=\langle A\rangle_0+\langle A\rangle_1+\langle A\rangle_2\quad\in\mathcal{G}_2=\mathcal{G}_2(\mathbb{R}).\qquad (5.161)\text{-}(5.162).$$

5.8.2.2. The 1-Spinor in the Euclidean Cartesian plane

First, we look at the even part of the plane algebra, as $E=\langle E\rangle_0+\langle E\rangle_2\ \in\mathcal{G}_2^+$

(5.401) $$E=\delta+\beta\boldsymbol{i}=\delta+\mathbf{B},\qquad \text{where } \mathbf{B}^2=-\beta^2,\ \text{and}\ \boldsymbol{i}^2=-1,$$

where the Euclidean bivector $\mathbf{B}$ is in the plane ***direction*** of $\boldsymbol{i}:=\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$, as the bivector span, $\mathbf{B}=\beta\boldsymbol{i}$.
The scalar $\delta\in\mathbb{R}$, multiplicative commute with $\mathbf{B}$, $\mathbf{B}\delta=\delta\mathbf{B}$, therefore by (5.389) $\exp(E)$

(5.402) $$\mathcal{Z}=e^E=e^{\delta+\beta\boldsymbol{i}}=e^{\delta}e^{\beta\boldsymbol{i}}=\rho e^{\beta\boldsymbol{i}}\quad\in\mathcal{G}_2^+,\ \text{with } \rho=e^{\delta}\in\mathbb{R},\ \beta\in\mathbb{R}, \text{and } \delta=\ln(\rho)\in\mathbb{R}.$$

Here in the plane, the power expansion in the even multivector E of max ***grade*** n=2 is

(5.403) $$\mathcal{Z}=e^E=1+\frac{E}{1!}+\frac{E^2}{2!}+\cdots+\frac{E^k}{k!}+\cdots$$

And the 1-spinor form as in section 5.2.9 (5.106) we have the ρ dilated Euler 1-rotor $U_\beta=\rho e^{\boldsymbol{i}\beta}$

(5.404) $$\mathbf{r}\mathbf{u}_1=\mathbf{r}\cdot\mathbf{u}_1+\mathbf{r}\wedge\mathbf{u}_1=\mathcal{Z}=\rho e^{\boldsymbol{i}\beta}=\rho\cos\beta+\boldsymbol{i}\rho\sin\beta,$$

where the last expression is the cartesian component coordinates to the multivector (5.138) using the polar coordinates (ρ,β) as input arguments, again the ***direction*** of this plane is $\boldsymbol{i}:=\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$.

5.8.2.3. The Lorentz 1-Spinor in the Line *Direction* Paravector Space

Second, we look at the simple mixed paravector algebra, as $\wp=\langle\wp\rangle_0+\langle\wp\rangle_1$ written as (5.240)

(5.405) $$\wp=\alpha 1+\zeta\mathbf{u}=\alpha+\mathbf{p},\qquad\text{where } \mathbf{p}^2=\zeta^2,\ \text{and}\ \mathbf{u}^2:=1.$$

In that $\alpha\in\mathbb{R}$, commute with $\mathbf{u}$ as $\alpha\mathbf{u}=\mathbf{u}\alpha$, we have the commuting product factors of $\exp(\wp)$

(5.406) $$\wp_\Lambda=e^{\wp}=e^{\alpha+\zeta\mathbf{u}}=e^{\alpha}e^{\zeta\mathbf{u}}=\rho_\Lambda e^{\zeta\mathbf{u}}.$$

The first factor in the last term is a dilation by a real scalar without ***direction***

(5.407) $$\rho_\Lambda=e^{\alpha}=\cosh\alpha+\sinh\alpha\quad\in\mathbb{R}.$$

The second factor is a Lorentz rotor paravector of the ***grade*** form $\langle A\rangle_0+\langle A\rangle_1$ (as (5.358))

(5.408) $$U_{\zeta,\mathbf{u}}^2=e^{\zeta\mathbf{u}}=\cosh(\zeta\mathbf{u})+\sinh(\zeta\mathbf{u})=\cosh\zeta+\mathbf{u}\sinh\zeta.$$

The reason for this is that $\mathbf{u}^2$=1, first the even expansion cancel the 1-vector ***direction***

(5.409) $$\cosh(\zeta\mathbf{u})=1+\frac{(\zeta\mathbf{u})^2}{2!}+\frac{(\zeta\mathbf{u})^4}{4!}+\cdots=1+\frac{\zeta^2}{2!}+\frac{\zeta^4}{4!}+\cdots=\cosh\zeta\quad\in\mathbb{R}.$$

Then the odd expansion separates the 1-vector ***direction***

(5.410) $$\sinh(\zeta\mathbf{u})=\zeta\mathbf{u}+\frac{(\zeta\mathbf{u})^3}{3!}+\frac{(\zeta\mathbf{u})^5}{5!}+\cdots=\mathbf{u}\left(\zeta+\frac{\zeta^3\mathbf{u}^2}{3!}+\frac{\zeta^5\mathbf{u}^4}{5!}+\cdots\right)=\mathbf{u}\sinh\zeta.$$

We find that (5.406) is a paravector as (5.242) containing a scalar part plus a 1-vector part

(5.411) $$\wp_\Lambda=e^{\alpha+\zeta\mathbf{u}}=\rho_\Lambda U_{\zeta,\mathbf{u}}=\rho_\Lambda\cosh\zeta+\mathbf{u}(\rho_\Lambda\sinh\zeta).$$

The exponential function $\exp(\wp)$ of a paravector is a paravector $\wp_\Lambda$ with preserved line ***direction***.