(5.290)
$$G = g_{11}P_+ + g_{12}\boldsymbol{\sigma}_1 P_- + g_{21}\boldsymbol{\sigma}_1 P_+ + g_{22}P_-$$
$$= ½(g_{11}+g_{22}) + ½(g_{12}+g_{21})\boldsymbol{\sigma}_1 + ½(g_{11}-g_{22})\boldsymbol{\sigma}_2 + ½(g_{12}-g_{21})\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1,$$

and its $\boldsymbol{\sigma}_1$-conjugation

(5.291)
$$G^{\boldsymbol{\sigma}_1} = g_{11}P_- + g_{12}\boldsymbol{\sigma}_1 P_+ + g_{21}\boldsymbol{\sigma}_1 P_- + g_{22}P_+$$
$$= ½(g_{11}+g_{22}) + ½(g_{12}+g_{21})\boldsymbol{\sigma}_1 - ½(g_{11}-g_{22})\boldsymbol{\sigma}_2 - ½(g_{12}-g_{21})\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1.$$

Then we have the form

(5.292) $$G = \alpha 1 + \nu_1\boldsymbol{\sigma}_1 + \nu_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \quad \text{and} \quad G^{\boldsymbol{\sigma}_1} = \alpha 1 + \nu_1\boldsymbol{\sigma}_1 - \nu_2\boldsymbol{\sigma}_2 - \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1.$$

5.6.1.4. An Example of a Matrix in $\mathcal{G}_2(\mathbb{R})$

A simple example is the special orthogonal rotation group $SO(2)$ of real 2×2 matrices of the type

(5.293) $$\begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix} = \begin{bmatrix}g_{11} & g_{12} \\ g_{21} & g_{22}\end{bmatrix}, \qquad \sim[e^{i\phi}],$$

We see that anti-symmetry cancel when $(g_{12}+g_{21}) = 0$ and $(g_{11}-g_{22}) = 0$, further

(5.294) $$½(g_{11}+g_{22}) = \alpha = \cos\phi \quad \text{and} \quad ½(g_{12}-g_{21}) = \beta = \sin\phi\,.$$

In this way, we get the 1-rotor form as (5.83) for a rotation

(5.295) $$G_{\text{rotor}} = U_\phi = \cos\phi + \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\sin\phi = e^{\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\phi},$$

and

(5.296) $$G^{\boldsymbol{\sigma}_1}_{\text{rotor}} = U^\dagger_\phi = \cos\phi - \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\sin\phi = e^{-\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\phi}.$$

This of course can be dilated by a factor ρ to a 1-spinor in the plane.

When we have $\nu_1 = ½(g_{12}+g_{21}) \neq 0$ and\or $\nu_2 = ½(g_{11}-g_{22}) \neq 0$, there is also involved some extension translation variation $\mathbf{t} = \nu_1\boldsymbol{\sigma}_1 + \nu_2\boldsymbol{\sigma}_2 \in \mathcal{G}_2(\mathbb{R})$ along the plane supported by $\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \in \mathcal{G}_2(\mathbb{R})$.

remember the unitarity (5.85) $U^\dagger_\theta U_\theta = U_\theta U^\dagger_\theta = 1$ for the 1-rotor in the geometric algebraic plane.

For the real $SO(2)$ matrix, we have the transposed

(5.297) $$\begin{bmatrix}g_{11} & g_{12} \\ g_{21} & g_{22}\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}g_{11} & g_{21} \\ g_{12} & g_{22}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{bmatrix} \qquad = [U^\dagger_\theta]$$

The product of these two (5.297) and (5.293)

(5.298) $$\begin{bmatrix}g_{11} & g_{12} \\ g_{21} & g_{22}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}g_{11} & g_{21} \\ g_{12} & g_{22}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix} = \cos^2\phi + \sin^2\phi = 1$$

Something similar for the determinant of (5.293) due to the anti-symmetry

(5.299) $$\begin{vmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{vmatrix} = \cos^2\phi + \sin^2\phi = 1$$

We say that the rotation matrix (5.293) is unitary.

Here we will not go further with the matrix formalism for the plane idea.

5.7. Plane Concept Idea of a Non-Euclidean Clifford Algebra

First, a natural 1-vector object $\mathbf{u}$ or $\mathbf{p} = \lambda_1\mathbf{u}$ can be drawn on a surface (paper) as an arrow for the intuition to indicate a natural ***direction*** $\mathbf{u}$ with extension magnitude $\mathbf{u}^2$=1 or $|\mathbf{p}| = |\lambda_1| \geq 0$. The fundamental idempotent multivector $½(1 \pm \mathbf{u})$ or the paravector $\wp = \lambda_0 + \mathbf{p}$ of the ***grade*** form $\langle A\rangle_0 + \langle A\rangle_1$ cannot be drawn direct for the intuition because the scalar part has no extension. To remedy this we will make use of the Minkowski space concept, inspired by [16], [17], [6].

5.7.1. Plane Geometric Clifford Algebra with Minkowski Signature for Measure Information

Besides the Euclidean plane concept $\mathcal{G}_{2,0}$ expressed in (5.198), we make a non-Euclidean plane $\mathcal{G}_{1,1}$. For this, we invent an external unit 1-vector γ_0 for the information development ***direction***, as a ***first grade quality*** (***pqg***-1) with a positive causal orientation towards the *future*.[268] We demand a positive signature Clifford metric (+) for this *causal* ***direction***

(5.300) $$\gamma_0^2 = +1, \qquad \text{and} \quad \widetilde{\gamma_0} = \gamma_0 = \overline{\gamma_0} = \gamma_0.$$

Conversely to this creative information ***direction*** γ_0, we invent unit 1-vectors γ_k for each one Descartes extension ***pqg***-1 ***directions***. For these ***directions***, we demand a negative signature (−)

(5.301) $$\gamma_k^2 = -1, \quad \text{for } k = 1,2,3, \quad \text{and} \quad \widetilde{\gamma_k} = \gamma_k, \text{ but } \overline{\gamma_k} = -\gamma_k.$$

The purpose of this is, that the quadratic norm for information development is in balance with the quadratic norm for the extension

(5.302) $$\boxed{\gamma_0^2 + \gamma_k^2 = 0}\,.$$

For each k the basis set $\{\gamma_0, \gamma_k\}$ is an orthonormal basis

(5.303) $$\gamma_0\cdot\gamma_k = 0 \quad \text{and} \quad |\gamma_0| = |\gamma_k| = 1,$$

for an abstract plane concept, we call it a $\mathcal{B}$-plane,[269] that has Clifford algebra $\mathcal{G}_{1,1}(\mathbb{R})$, signatures (+, −).

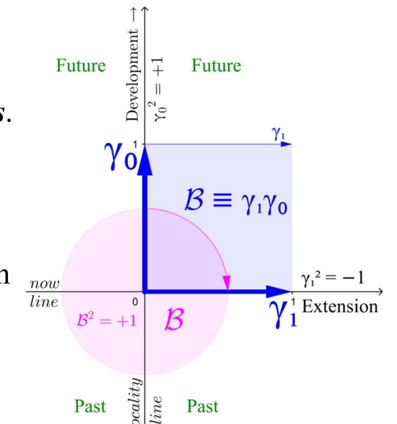

Figure 5.49 The $\underline{\mathcal{B}\text{-bivector}}$ $\mathcal{B} \coloneqq \gamma_1\gamma_0$, ($k$=1) forming a $\underline{\mathcal{B}\text{-plane}}$ from the 1-vectors γ_0 for <u>development</u> with positive signature γ_0^2 =1, and for <u>extension</u> γ_1 with antagonist signature γ_1^2=−1. Forming any unit $\mathcal{B}$-bivector amoeba $\mathcal{B} = \mathcal{B}$, in $\mathcal{B}$-plane, with signature $\mathcal{B}^2$=1. This intuit display *object* is an abstraction of measure *substance* of information about the extension.

From this abstraction of this 1-vector basis $\{\gamma_0, \gamma_1\}$ we form a mix of two new units

(5.304) $$1 \coloneqq \gamma_0\gamma_0 = \gamma_0^2 = +1, \qquad \text{the real scalar unit.}$$

(5.305) $$\mathcal{B} \coloneqq \gamma_1\gamma_0 = \gamma_1\wedge\gamma_0, \qquad \text{the } \mathcal{B}\text{-plane unit pseudoscalar } \mathcal{B}\text{-bivector, with the reversion}$$

(5.306) $$\overline{\mathcal{B}} = \widetilde{\mathcal{B}} = -\mathcal{B} = \gamma_0\gamma_1 = \gamma_0\wedge\gamma_1, \qquad \text{in that} \quad \gamma_1\wedge\gamma_0 = -\gamma_0\wedge\gamma_1$$

For the signature square of this $\mathcal{B}$-plane pseudoscalar $\mathcal{B}$-bivector unit (5.305) we have

(5.307) $$\mathcal{B}^2 = 1 \qquad = \gamma_1\gamma_0\gamma_1\gamma_0 = -\gamma_1\gamma_1\gamma_0\gamma_0 = 1.$$

The geometric *substance* structure of the $\mathcal{B}$-bivector ***direction*** plane is displayed in Figure 5.49. From the defining basis $\{\gamma_0, \gamma_1\}$, we form the mixed basis $\{1, \mathcal{B}\}$, a *scalar* and a *pseudoscalar* unit. This we combine to a full mixed basis for the <u>Minkowski $\mathcal{B}$-plane algebra</u> $\mathcal{G}_{1,1}(\mathbb{R})$

(5.308) $$\{1,\ \gamma_0,\ \gamma_1,\ \mathcal{B} \coloneqq \gamma_1\gamma_0\}.$$

The action of the $\mathcal{B}$ multiplication operations give the exchange properties

(5.309) $$\boxed{\begin{array}{lll}\mathcal{B}\gamma_0 = \gamma_1, & \mathcal{B}\gamma_1 = \gamma_0, & \mathcal{B}^2 = 1, \\ \gamma_0\mathcal{B} = -\gamma_1, & \gamma_1\mathcal{B} = -\gamma_0. & 1 \in \mathbb{R}, \quad \text{is the neutral multiplication identity.}\end{array}}$$

[268] This ***primary quality of first grade*** as a ***direction*** towards the future has no Descartes extension. It is a ***quality*** of counting times of occurrence in a process of development; one count is the unit 1-vector γ_0, with γ_0^2=1 for *FORWARD*. We use $\tau\gamma_0, \tau\in\mathbb{R}$. – In a tradition of classical mechanics, this count is often interpreted as a continuous floating river of time. (a mysterious concept.)

[269] The name $\mathcal{B}$-plane is used instead of the obvious name Minkowski-plane to prevent confusion to other conceptual interpretations.