5.6.1.3. The Matrices of the Geometric Algebra $\mathcal{G}_2(\mathbb{R})$

How can matrices represent the ***direction*** concept just as geometric multivectors do?
We recall (5.198) the plane space concept supported by the mixed 2^2-dimensional standard basis

(5.271) $\{1,\ \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2,\ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\},$ $\quad (\,\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\,)$

The general form of a plane geometric multivector (5.197) is written

(5.272) $G = \ \alpha 1 + \nu_1\boldsymbol{\sigma}_1 + \nu_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \quad \in \mathcal{G}_2(\mathbb{R}), \qquad$ where $\alpha, \nu_1, \nu_2, \beta \in \mathbb{R}$.

The 1-vectors $\nu_k\boldsymbol{\sigma}_k$ mutual anticommute, as with the pseudoscalar bivector $\beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ too, but all scalars commute with all elements. We prerequisite $\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = 1$ and $\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_1 = 0$.

As a complement to the cartesian form[267] $\mathbf{x} = \nu_1\boldsymbol{\sigma}_1 + \nu_2\boldsymbol{\sigma}_2$ supported by $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$ we introduce from the paravector idea the mixed 2-tuple matrices as column or row editions

(5.273) $\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix}$ or $(1\ \ \boldsymbol{\sigma}_1)$, together with $\begin{pmatrix}1\\ \boldsymbol{\sigma}_2\end{pmatrix}$ or $(1\ \ \boldsymbol{\sigma}_2)$.

We take the first two 2-tuple forms as our master and ignore implicit the last two, and Instead, use the idempotent projection form (5.235) for this projected ***direction***

(5.274) $\boxed{P_+ = ½(1+\boldsymbol{\sigma}_2)}$ and its Clifford conjugated $\widetilde{P_+} = \overline{P_+} = \boxed{P_- = ½(1-\boldsymbol{\sigma}_2)}$.

These two conjugated are mutual annihilating $P_+P_- = 0$. (Refer to (5.247) and (5.223))
Inspired by the reflection in a ***direction*** 1-vector § 5.4.2.1 we will as in [14] p.79 let P_+ act on the row form $(1\ \ \boldsymbol{\sigma}_1)$ and then let $(1\ \ \boldsymbol{\sigma}_1)^{\mathrm{T}}$ left act on this result in a canonical way

(5.275) $$\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix}P_+(1\ \ \boldsymbol{\sigma}_1) = \begin{pmatrix}P_+\\ \boldsymbol{\sigma}_1P_+\end{pmatrix}(1\ \ \boldsymbol{\sigma}_1) = \begin{bmatrix}P_+ & P_+\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_1P_+ & \boldsymbol{\sigma}_1P_+\boldsymbol{\sigma}_1\end{bmatrix} = \begin{bmatrix}P_+ & \boldsymbol{\sigma}_1P_-\\ \boldsymbol{\sigma}_1P_+ & P_-\end{bmatrix},$$

to get a 2×2 matrix form for a natural plane *geometric basis*, we as Sobczyk call a ***spectral basis***. This $\mathcal{G}_2(\mathbb{R})$ plane matrix basis represents the support of $4 = 2^2$-dimensions of real numbers similar to the geometric algebraic form (5.272), but it will have different properties that we certainly not called coordinates but matrix elements, e.g. $g_{11}, g_{12}, g_{21}, g_{22} \in \mathbb{R}$.
From this idea, we will map the geometric multivector G (5.272) to a 2×2 real matrix

(5.276) $$G \longrightarrow [G] = \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$$

Having the matrix $[G]$ we seek G by the map $[G] \longrightarrow G$. We use the reversed matrix multiplication operation to (5.275) on $[G]$

(5.277) $$G = \overbrace{(1\ \ \boldsymbol{\sigma}_1)P_+}\,\underbrace{[G]}\,\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix} = \overbrace{(P_+\ \ \boldsymbol{\sigma}_1P_+)}\underbrace{\begin{pmatrix}g_{11}+g_{12}\boldsymbol{\sigma}_1\\ g_{21}+g_{22}\boldsymbol{\sigma}_1\end{pmatrix}} = g_{11}P_+ + g_{12}\boldsymbol{\sigma}_1P_- + g_{21}\boldsymbol{\sigma}_1P_+ + g_{22}P_-,$$

where we use the fact: $P_- = \boldsymbol{\sigma}_1P_+\boldsymbol{\sigma}_1$, and $P_+\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1P_-$, or $\boldsymbol{\sigma}_1P_+ = P_-\boldsymbol{\sigma}_1$.
By $\boldsymbol{\sigma}_1^2=1$ we have the unitarity

(5.278) $$(1\ \ \boldsymbol{\sigma}_1)P_+\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix} = (1\ \ \boldsymbol{\sigma}_1)\begin{pmatrix}P_+\\ P_+\boldsymbol{\sigma}_1\end{pmatrix} = P_+ + \boldsymbol{\sigma}_1P_+\boldsymbol{\sigma}_1 = P_+ + P_- = 1.$$

To transform the geometric multivector (5.272) $G = \alpha 1 + \nu_1\boldsymbol{\sigma}_1 + \nu_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ we will as Garret Sobczyk [14] p.80, [15] multiply on both sides with this unit (5.278) to get the matrix form

(5.279) $$\underset{\smile}{G} = (1\ \ \boldsymbol{\sigma}_1)P_+\overbrace{\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix}\underset{\smile}{G}\,(1\ \ \boldsymbol{\sigma}_1)}P_+\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix} = (1\ \ \boldsymbol{\sigma}_1)P_+\underbrace{\overbrace{\begin{bmatrix}G & G\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_1G & \boldsymbol{\sigma}_1G\boldsymbol{\sigma}_1\end{bmatrix}}P_+}\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix}$$

By (5.281) resulting in: $\qquad = G = \quad (1\ \ \boldsymbol{\sigma}_1)P_+\overbrace{\underbrace{\begin{bmatrix}\alpha+\nu_2 & \nu_1+\beta\\ \nu_1-\beta & \alpha-\nu_2\end{bmatrix}}}\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix}.$

[267] Here we use the scale coefficients ν_k for the line extension variation $\nu_k\boldsymbol{\sigma}_k$ instead of $x_k\boldsymbol{\sigma}_k$ for the position coordinate x_k.

This result, we get for each place in the matrix $\frown$ by use of the form $\smile$ and the fact that

(5.280) $P_+\boldsymbol{\sigma}_1P_+ = 0, \quad P_+\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1P_+ = 0, \quad$ and $\quad P_+\boldsymbol{\sigma}_2 = P_+1.$

Then

(5.281) $$\left\{\begin{aligned}
P_+\,\overset{\frown}{G}\,P_+ &= P_+(\alpha + \nu_1\boldsymbol{\sigma}_1 + \nu_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1)P_+ &&= P_+(\alpha+\nu_2)\\
P_+\,\overset{\frown}{G\boldsymbol{\sigma}_1}\,P_+ &= P_+(\alpha + \nu_1\boldsymbol{\sigma}_1 + \nu_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1)\boldsymbol{\sigma}_1P_+ &&= P_+(\nu_1+\beta)\\
P_+\,\overset{\frown}{\boldsymbol{\sigma}_1G}\,P_+ &= P_+\boldsymbol{\sigma}_1(\alpha + \nu_1\boldsymbol{\sigma}_1 + \nu_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1)P_+ &&= P_+(\nu_1-\beta)\\
P_+\,\overset{\frown}{\boldsymbol{\sigma}_1G\boldsymbol{\sigma}_1}\,P_+ &= P_+\boldsymbol{\sigma}_1(\alpha + \nu_1\boldsymbol{\sigma}_1 + \nu_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1)\boldsymbol{\sigma}_1P_+ &&= P_+(\alpha-\nu_2)
\end{aligned}\right\}$$

A plane geometric algebraic element $G \in \mathcal{G}_2(\mathbb{R})$ (5.272) is by the *projection spectral basis* (5.275) mapped to a real matrix as (5.276)

(5.282) $$G \longrightarrow [G] = \begin{bmatrix}\alpha+\nu_2 & \nu_1+\beta\\ \nu_1-\beta & \alpha-\nu_2\end{bmatrix}$$

How will this look when we do not specify the coefficients in (5.272) and just want to find an expression from an abstract multivector $G \in \mathcal{G}_2(\mathbb{R})$ for a matrix $[G]$ representing the same plane ***quality*** of our physical ***entity***? First, we do not know the expansion (5.272), then:
For the projector P_+ for the form expressed in (5.277) will be stuck in the first line of (5.279) at

(5.283) $$G = (1\ \ \boldsymbol{\sigma}_1)P_+[G]\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix} = (1\ \ \boldsymbol{\sigma}_1)\underbrace{P_+\begin{bmatrix}G & G\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_1G & \boldsymbol{\sigma}_1G\boldsymbol{\sigma}_1\end{bmatrix}P_+}\begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix}.$$

From this, we extract that

(5.284) $$P_+[G] = P_+\begin{bmatrix}G & G\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_1G & \boldsymbol{\sigma}_1G\boldsymbol{\sigma}_1\end{bmatrix}P_+$$

But we can as well use the projection P_- in our deduction. The change is that we go into a conjugated picture by an *inner automorphism*. We as Sobczyk [14], [15] invent a $\boldsymbol{\sigma}_1$-conjugation

(5.285) $G^{\boldsymbol{\sigma}_1} = \boldsymbol{\sigma}_1G\boldsymbol{\sigma}_1 \quad \Rightarrow \quad P_+^{\boldsymbol{\sigma}_1} = \boldsymbol{\sigma}_1P_+\boldsymbol{\sigma}_1 = P_-$

By this, we transform (5.284) to

(5.286) $$P_-[G^{\boldsymbol{\sigma}_1}] = P_-\begin{bmatrix}\boldsymbol{\sigma}_1G\boldsymbol{\sigma}_1 & \boldsymbol{\sigma}_1G\\ G\boldsymbol{\sigma}_1 & G\end{bmatrix}P_-.$$

We seek the same real matrix of the form (5.276), therefore we presume $[G^{\boldsymbol{\sigma}_1}] = [G] = \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$, with $g_{kj} \in \mathbb{R}$ for the real matrix $[G]$. By the unitarity (5.278) $P_+ + P_- = 1$ we form the sum

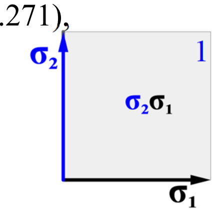
(5.287) $$P_+[G] + P_-[G^{\boldsymbol{\sigma}_1}] = (P_+ + P_-)[G] = [G].$$

This sum of (5.284) and (5.286) then give us the desired real matrix

(5.288) $$[G] = P_+\begin{bmatrix}G & G\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_1G & \boldsymbol{\sigma}_1G\boldsymbol{\sigma}_1\end{bmatrix}P_+ + P_-\begin{bmatrix}\boldsymbol{\sigma}_1G\boldsymbol{\sigma}_1 & \boldsymbol{\sigma}_1G\\ G\boldsymbol{\sigma}_1 & G\end{bmatrix}P_-.$$

This is the map of the unspecified abstract geometric algebraic multivector $G \in \mathcal{G}_2(\mathbb{R})$ supported from the ***spectral basis*** (5.275), given from our intuitive object standard basis (5.271), using the mutual annihilating projection operators (5.274)

(5.289) $\begin{bmatrix}P_+ & \boldsymbol{\sigma}_1P_-\\ \boldsymbol{\sigma}_1P_+ & P_-\end{bmatrix}, \quad \{1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\}, \quad \begin{pmatrix}1\\ \boldsymbol{\sigma}_1\end{pmatrix}, \quad P_\pm = ½(1 \pm \boldsymbol{\sigma}_2).$



The real matrix form $[G] = \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$ we cannot intuit as a geometric object on a natural surface in space, but we write it out as