From all this, we achieve the spectral projective properties

(5.248) $\wp P_u = P_u\wp = \lambda_+ P_u$, and $\wp\overline{P}_u = \overline{P}_u\wp = \lambda_-\overline{P}_u$, together with

(5.249) $\overline{\wp}P_u = P_u\overline{\wp} = \lambda_- P_u$, and $\overline{\wp}\,\overline{P}_u = \overline{P}_u\overline{\wp} = \lambda_+\overline{P}_u$.

(5.250) $\lambda_+\lambda_- = (\lambda_0+\lambda_u)(\lambda_0-\lambda_u) = \lambda_0^2-\lambda_u^2 \in\mathbb{R}$, is a scalar given from the components (5.245).

The product of two paravectors $\wp = \lambda_+P_u+\lambda_-\overline{P}_u$ and $\wp' = \lambda'_+P_u+\lambda'_-\overline{P}_u$ is

(5.251) $\wp\wp' = (\lambda_+P_u+\lambda_-\overline{P}_u)(\lambda'_+P_u+\lambda'_-\overline{P}_u) = (\lambda_+\lambda'_+)P_u+(\lambda_-\lambda'_-)\overline{P}_u$.

The paravector auto-product square is

(5.252) $\wp\wp = (\lambda_+P_u+\lambda_-\overline{P}_u)(\lambda_+P_u+\lambda_-\overline{P}_u) = \lambda_+^2P_u+\lambda_-^2\overline{P}_u$.

What about the observable magnitude of a paravector?
We evaluate the basis auto product square $P_u^2 = P_u$ it is certainly not a scalar for the magnitude.
Instead, we will use the Clifford conjugation quadratic form by using the scalar (5.250)

(5.253) $\wp\overline{\wp} = (\lambda_+P_u+\lambda_-\overline{P}_u)(\lambda_+\overline{P}_u+\lambda_-P_u) = \lambda_+\lambda_-(P_u+\overline{P}_u) = \lambda_+\lambda_- = \lambda_0^2-\lambda_u^2 \in\mathbb{R}$.

This is in full agreement with the product of (5.240) and (5.241) in the definition basis {1, u}

(5.254) $\wp\overline{\wp} = \overline{\wp}\wp = \lambda_0^2-\mathrm{p}^2 = (\lambda_0+\mathrm{p})(\lambda_0-\mathrm{p}) = (\lambda_0 1+\lambda_u\mathrm{u})(\lambda_0 1-\lambda_u\mathrm{u}) = \lambda_0^2-\lambda_u^2 \in\mathbb{R}$,

positive, null, or negative, that yields the *observable pure magnitude for the paravector*

(5.255) $|\wp|_0 = |\overline{\wp}|_0 = \sqrt{|\lambda_0^2-\lambda_u^2|} \geq 0$.

From this, we conclude that the projection basis $\{P_u,\overline{P}_u\}$ consists of *null* basis vectors

(5.256) $P_u\overline{P}_u = 0 \implies |P_u|_0 = |\overline{P}_u|_0 = 0$

This (5.254) indicates a Clifford algebra structure with a Minkowski signature metric.
– To immediately continue with Minkowski metric jump to section 5.7.

5.5.4.4. Non Measurable Fictive Magnitude of Paravectors

When we follow the definition of multivector magnitude given by Hestenes [10]p.46 as (5.166)-(5.167) for the paravector $\wp = \lambda_0 1+\lambda_u\mathrm{u} = \langle\wp\rangle_0+\langle\wp\rangle_1 = \langle\lambda_0 1\rangle_0+\langle\lambda_u\mathrm{u}\rangle_1$ we have

(5.257) $|\wp|^2 = |\langle\lambda_0 1\rangle_0|^2+|\langle\lambda_u\mathrm{u}\rangle_1|^2 = \lambda_0^2+\lambda_u^2$.

This is a simple pure mathematic defined magnitude.
Hestenes as in [13] p.13(1.51) made the simple assumption that $\langle A^\dagger A\rangle_0 \geq 0$. (5.257)
The question is, will this paravector idea of magnitude be measurable in physics?

When it comes to Clifford conjugation $\langle\widetilde{A}\,A\rangle_0$ and special parity inversion $\langle\overline{A}\,A\rangle_0$ we have the simple reality that for 1-vector $\langle\overline{\mathrm{p}}\mathrm{p}\rangle_0 = -|\mathrm{p}|^2$ and for the unit $\langle\overline{\mathrm{u}}\,\mathrm{u}\rangle_0 = \overline{\mathrm{u}}\,\mathrm{u} = -\mathrm{u}^2 = -1$, but we know that the Euclidian 1-vector has as $\mathrm{u}^2 = 1$ the magnitude $\langle\mathrm{pp}\rangle^{½} = \langle\mathrm{p}^\dagger\mathrm{p}\rangle^{½} = |\mathrm{p}| = |\lambda_u|$, that is independent of the scalar ***quality*** λ_0 magnitude $|\lambda_0|$.

This *observable pure magnitude* measure of an object 1-vector has an extension ***direction***, though its founded in the idea of the formulation in (5.254)-(5.255). But here the outstanding question is, what ***direction*** count does the pure scalar represents in physics? Will the quantity $\sqrt{\lambda_0^2+\lambda_u^2}$ appear in reality? It is obvious that it is not an intuitive object per se.
The simplest way is to demand the synthetic judgment $\lambda_0^2 \geq \lambda_u^2 \implies |\lambda_0| \geq |\mathrm{p}|$

(5.258) $|\wp|_0^2 \to \langle\wp\overline{\wp}\rangle_0 = \lambda_0^2-\lambda_u^2 \geq 0$

This will be treated further below in section 5.7 and later in chapter III. 7.

Anyway, when it comes to geometric algebraic bivectors the reversion product is positive $\langle\widetilde{\mathrm{B}}\,\mathrm{B}\rangle_0 = \langle\mathrm{B}^\dagger\mathrm{B}\rangle_0 = \mathrm{B}^\dagger\mathrm{B} \geq 0$.

5.6. The Real Matrix Representation for the Plane Concept

5.6.1. The Fundamentals of Matrices in a Plane Algebra $\mathcal{G}_2$

5.6.1.1. Matrices for a Cartesian 1-vector Concept for a Euclidean plane $\mathbb{R}_1^2$

The orthonormal Cartesian basis set we name $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\}$ as the ***directions*** in space for our plane.
These objects we associate with 2-tuples, first as columns, next as rows

(5.259) $\boldsymbol{\sigma}_1 \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$, $\boldsymbol{\sigma}_2 \leftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$, and $\boldsymbol{\sigma}_1 \leftrightarrow (1\ \ 0)$, $\boldsymbol{\sigma}_2 \leftrightarrow (0\ \ 1)$.

The abstract 2-tuple basis is then with the geometric ***directions*** implicit hidden

(5.260) $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\} \leftrightarrow \left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\} \leftrightarrow \{(1\ \ 0),(0\ \ 1)\}$

A general Cartesian 1-vectors in the plane we express as (5.136)

(5.261) $\mathbf{x} = x_1\boldsymbol{\sigma}_1+x_2\boldsymbol{\sigma}_2$, with $x_k \in\mathbb{R}$, from the perpendicular basis objects $\boldsymbol{\sigma}_2\perp\boldsymbol{\sigma}_1$.

As 2-tuples, first as columns,[265] next as rows

(5.262) $\mathbf{x} \leftrightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x^1\\x^2\end{pmatrix}$ and $\mathbf{x} \leftrightarrow (x_1\ \ x_2)$, where $(x_1\ \ x_2)^{\mathrm{T}} = \begin{pmatrix}x^1\\x^2\end{pmatrix}$.

Then the coordinates are performed as an inner product[266] of such matrix 2-tuples, row·column

(5.263) $x_1 = \mathbf{x}\cdot\boldsymbol{\sigma}_1 = \mathbf{x}\cdot\begin{pmatrix}1\\0\end{pmatrix} = (x_1\ \ x_2)\cdot\begin{pmatrix}1\\0\end{pmatrix}$ and $x_2 = \mathbf{x}\cdot\boldsymbol{\sigma}_2 = \mathbf{x}\cdot\begin{pmatrix}0\\1\end{pmatrix} = (x_1\ \ x_2)\cdot\begin{pmatrix}0\\1\end{pmatrix}$.

Orthonormality is expressed as

(5.264) $\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 = (0\ \ 1)\cdot\begin{pmatrix}1\\0\end{pmatrix} = (1\ \ 0)\cdot\begin{pmatrix}0\\1\end{pmatrix} = 0$, and e.g. $\boldsymbol{\sigma}_1^2 = (1\ \ 0)\cdot\begin{pmatrix}1\\0\end{pmatrix} = 1$.

From (5.261)-(5.262) we define two arbitrary 1-vectors in the $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\}$ plane

(5.265) $\mathbf{a} \leftrightarrow \begin{pmatrix}a^1\\a^2\end{pmatrix} = a^j$ and $\mathbf{b} \leftrightarrow (b_1\ \ b_2) = b_k$, for $j,k = 1{,}2$.

We form the product of the two matrix 2-tuples and see the differences
First, the inner scalar product is the same as for the 1-vectors $\mathbf{b}\cdot\mathbf{a} \in\mathbb{R}$

(5.266) $(b_1\ \ b_2)\cdot\begin{pmatrix}a^1\\a^2\end{pmatrix} = b_1a^1+b_2a^2 = b_ka^k \in\mathbb{R}$.

But we have a different tuple epistemology than the geometric products of these 1-vectors **ba** or **ab** as above (5.56), when we form the product of the matrix 2-tuples to a 2×2 matrix as

(5.267) $\begin{pmatrix}a^1\\a^2\end{pmatrix}(b_1\ \ b_2) = \begin{bmatrix}a^1b_1 & a^1b_2\\ a^2b_1 & a^2b_2\end{bmatrix} = a^jb_k = \begin{bmatrix}c^1{}_1 & c^1{}_2\\ c^2{}_1 & c^1{}_2\end{bmatrix} = c^j{}_k$, for $j,k = 1{,}2$.

5.6.1.2. Examples of Matrices of Geometric Multivectors

In a Euclidean-Cartesian plane, we define a set of 1-vectors

(5.268) $\mathbf{a}_j = a_{1,j}\boldsymbol{\sigma}_1+a_{2,j}\boldsymbol{\sigma}_2 = a^k{}_j\boldsymbol{\sigma}_k \in\mathcal{G}_2(\mathbb{R})$, with $a_{k,j} = a^k{}_j \in\mathbb{R}$, k=1,2, for Cartesian plane.

From this, for $m\in\mathbb{N}$ we form a m-row matrix

(5.269) $[\mathbf{a}]_{(m)} = (\mathbf{a}_1,\mathbf{a}_2,\ldots\mathbf{a}_m) \leftrightarrow \begin{bmatrix}a^1{}_1 & a^1{}_2 & \cdots & a^1{}_m\\ a^2{}_1 & a^2{}_2 & \cdots & a^2{}_m\end{bmatrix}$.

From this, we form its transposed m-column matrix

(5.270) $[\mathbf{a}]^{\mathrm{T}}_{(m)} = (\mathbf{a}_1,\mathbf{a}_2,\ldots\mathbf{a}_m)^{\mathrm{T}} = [\mathbf{a}]^{(m)} = \begin{pmatrix}\mathbf{a}^1\\ \mathbf{a}^2\\ \vdots\\ \mathbf{a}^m\end{pmatrix} \leftrightarrow \begin{bmatrix}a_1{}^1 & a_2{}^1\\ a_1{}^2 & a_2{}^2\\ \vdots & \vdots\\ a_1{}^m & a_2{}^m\end{bmatrix} \sim \begin{bmatrix}a_{1,1} & a_{2,1}\\ a_{1,2} & a_{2,2}\\ \vdots & \vdots\\ a_{1,m} & a_{2,m}\end{bmatrix}$.

[265] Generally upper index x^k is used for *column* numbers to indicate contravariance opposite covariance or *row* numbers x_k. In the orthogonal case, there is no difference $x^k = x_k$ (do not mistake k in x^k for exponents).
[266] We presume the reader is familiar with the rules of matrix multiplication. For this, I here have recalled it from [14] Chapter 4.

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