5.5.4. The Idempotent Operation

Operation by a multivector M can have some impact on an ***entity*** in physics. An extra operation by the same multivector M in some situations has no new effect.
This opportunity is called idempotence and for multivector-operations expressed as $M(M) = MM = M^2 = M$. The idempotent operation is often called a projection P where

(5.228) $P(P) = PP = P^2 = P.$

A projection multivector written in components for the plane basis $\{1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\}$ (5.198) is

(5.229) $P = \alpha + x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1,$

this is for the demand of idempotence

(5.230) $P^2 = (\alpha^2 + x_1^2 + x_2^2 - \beta^2) + (2\alpha x_1)\boldsymbol{\sigma}_1 + (2\alpha x_2)\boldsymbol{\sigma}_2 + (2\alpha\beta)\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \alpha + x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = P.$

The scalar part must be preserved, as well as each independent ***direction***

(5.231) $(\alpha^2 + x_1^2 + x_2^2 - \beta^2) = \alpha, \quad (2\alpha x_1) = x_1, \quad (2\alpha x_2) = x_2, \quad (2\alpha\beta) = \beta.$

The over simplest solution $\alpha = 0$ demand $x_1 = x_2 = \beta = 0$ is nothing.
The obvious possibility for idempotence is

(5.232) $\alpha = ½ \quad \text{and} \quad x_1^2 + x_2^2 - \beta^2 = ¼ \quad \Rightarrow \quad \beta = \pm\sqrt{x_1^2 + x_2^2 - ¼}\,.$

Then we have all the possibilities of projection operators

(5.233) $P = ½ + x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 \pm \sqrt{x_1^2 + x_2^2 - ¼}\,\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1.$

Again, using the Cartesian 1-vector $\mathbf{x} = x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2$ we get for all arbitrary 1-vector $\mathbf{x}$, with $|\mathbf{x}| \geq ½$ in the plane ***direction*** $\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ the idempotent projection element

(5.234) $P = (½ + \mathbf{x}) \pm \boldsymbol{i}\sqrt{\mathbf{x}^2 - ¼}\,.$

Taking the arbitrary unit vector ***direction*** u, with $\mathrm{u}^2 = 1 \Rightarrow |\mathrm{u}| = 1$ and setting $\mathbf{x} = \pm½\mathrm{u}$ in (5.234) we get writ of the plane part $\boldsymbol{i}$ and have the simplest primary idempotent projection operators

(5.235) $P_\mathrm{u} = P_+ = ½(1 + \mathrm{u}), \quad$ and its Clifford conjugated $\quad \overline{P}_\mathrm{u} = \widetilde{P}_\mathrm{u} = P_- = ½(1 - \mathrm{u}),$

for a unit $\mathrm{u}^2 = 1$ ***direction*** 1-vector u in space, independent of any specific plane. – Anyway, defined generally $\mathrm{u} \neq \pm 1$, this unit is indeed not a member of any scalar field $\mathrm{u} \notin \mathbb{R}, \mathbb{C}$, or $\mathbb{K}$, but in our primitive definition, u is certainly a member of a 1-vector field with a division algebra.
Do we restrict the ***maximal grade to one*** for the algebra form $\langle A\rangle_0 + \langle A\rangle_1$ we can consider $\mathrm{u} \sim \langle A\rangle_1$ as the pseudoscalar $\mathrm{u}^2 = 1$ of this algebra, with no reverse orientation, but the Clifford conjugated[261]

(5.236) $\overline{\mathrm{u}} = -\mathrm{u}, \quad$ with $\quad \overline{\mathrm{u}}^2 = 1$

The primary idempotent member P_u of the multivector algebra of grades $\langle A\rangle_0 + \langle A\rangle_1$ *has no inverse*, and any real scaled dilation αP_u with idempotence $(\alpha P_\mathrm{u})^2 = \alpha^2 P_\mathrm{u}$ *has no inverse* $(\alpha P_\mathrm{u})^{-1}$, because:

(5.237) $(1 + \mathrm{u})^{-1} = \frac{1}{1 + \mathrm{u}} = \frac{1}{(1 + \mathrm{u})}\frac{(1 - \mathrm{u})}{(1 - \mathrm{u})} = \frac{1 - \mathrm{u}}{1^2 - \mathrm{u}^2} = \frac{1 - \mathrm{u}}{1 - 1} = \frac{1 - \mathrm{u}}{0}, \quad$ that is undefined.

These elements αP_u is therefore not members of an associative division algebra.

Multiplying a dilated primary idempotent projection operator by its own ***direction*** unit 1-vector make no change to the projection operator

(5.238) $\alpha P_\mathrm{u}\mathrm{u} = \mathrm{u}\alpha P_\mathrm{u} = \alpha P_\mathrm{u} \quad = ½\alpha(1 + \mathrm{u})\mathrm{u} = ½\,\alpha(\mathrm{u} + 1), \quad \alpha \in \mathbb{R}.$

We say $\alpha P_\mathrm{u} = ½\alpha(1 + \mathrm{u})$ has the ***quality*** of absorbing factors of its own ***direction*** unit u.
To count u as a pseudoscalar we need to restrict the algebra to one geometric line ***direction*** towards infinity. I.e., any proper 1-vector $\mathbf{p} \sim \langle A\rangle_1$ has to fulfil $\mathbf{p} \wedge \mathrm{u} = 0$, which results in a line span

(5.239) $\mathbf{p} = \lambda_1\mathrm{u}, \quad$ for $\forall\lambda_1 \in \mathbb{R}, \quad$ from basis $\{\mathrm{u}\}$.

The scalar 1 in (5.235) is defined as the magnitude of u, $|\mathrm{u}| = \mathrm{u}^2 = 1$. See co-linear concept §4.4.4.1.

[261] For ***grade*** 1-vectors the Clifford conjugated is the same as the *parity inversion* $\tilde{\mathrm{u}} = \overline{\mathrm{u}} = -\mathrm{u}$, in that $\mathrm{u}^\dagger = \mathrm{u}$ for reversion.

5.5.4.2. The Paravector Concept

The full mixed standard basis for multivectors of grade form $\langle A\rangle_0 + \langle A\rangle_1$, then has the form $\{1, \mathrm{u}\}$.
In this basis, we span multivectors of ***grades*** ≤ 1 we call *paravectors*[262] expressed in the form

(5.240) $\wp = \lambda_0 1 + \lambda_\mathrm{u}\mathrm{u} \quad = \lambda_0 + \mathbf{p},$

where $\mathbf{p} = \lambda_1\mathrm{u} = \lambda_1\hat{\mathbf{p}}$, and we use the real component coordinates $\lambda_0, \lambda_1 \in \mathbb{R}$.
This paravector has a Clifford conjugated (or here just first ***grade*** *parity inversion*)

(5.241) $\overline{\wp} = \lambda_0 1 - \lambda_\mathrm{u}\mathrm{u} \quad = \lambda_0 - \mathbf{p}.$

Such a multivector *subject* consist of a substance of a ***direction*** as a ***primary quality of first grade*** (***pqg***-1) together with a scalar as a ***primary quality of zero grade*** (***pqg***-0) with ***no-direction*** as a surplus scalar para idea to the Descartes extension ***pqg***-1 idea.
As an object we have an extended vector $\mathbf{p} = \lambda_\mathrm{u}\mathrm{u}$ that can be drawn on a surface as an arrow for intuition, to analogue indicate a natural ***direction*** u with an extension magnitude $|\mathbf{p}| = |\lambda_1| \geq 0$.
But we do not have any intuition object of the scalar part λ_0. From the object unit $\mathrm{u}^2 = 1$ ***direction*** u we understand the geometric linear ***direction*** (*parity inversion*) $\overline{\mathbf{p}} = -\mathbf{p}$ of orientation for the Clifford conjugation. For intuit interpretation consult drawings above at § 4.4.2.5 at (4.60)-(4.61).
Of course, the paravector can exist in higher dimensions, e.g. for a plane $\mathbf{p} = \lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2$ implying the paravector $\wp = \lambda_0 + \mathbf{p} = \lambda_0 1 + \lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2$ from a basis $\{1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$, but this basis implies further the $\langle A\rangle_2$ pseudoscalar $\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ as (5.198) which we *here* try to avoid.[263]
Therefore, we settle from the simple basis $\{1, \mathrm{u}\}$ to make the fundamental structure for starting ***direction*** as a ***primary quality of first grade*** (***pqg***-1) loud and clear.[264]

- Anyway, alternative we remember Immanuel Kant 1768 [11] p.361-72, (der Gegenden im Raume).

5.5.4.3. The Projection of a Paravector on Its Idempotent Basis

The paravector structure is also, as by Garret Sobczyk [14] Chap.2, called hyperbolic numbers.
The defining foundation of the paravector ide is the basis $\{1, \mathrm{u}\}$ that supports each paravector as

(5.242) $\wp = \lambda_0 1 + \lambda_\mathrm{u}\mathrm{u}, \quad$ and its Clifford conjugated $\quad \overline{\wp} = \lambda_0 1 - \lambda_\mathrm{u}\mathrm{u}$

For investigating the structure of this paravector idea we will reformulate it in the *idempotent* orthogonal basis $\{P_\mathrm{u}, \overline{P}_\mathrm{u}\}$ defined from (5.235) as the *simplest primary idempotent **quality** basis set*

(5.243) $P_\mathrm{u}^2 = P_\mathrm{u} = ½(1 + \mathrm{u}), \quad$ and its Clifford conjugated $\quad \overline{P}_\mathrm{u}^2 = \overline{P}_\mathrm{u} = ½(1 - \mathrm{u}).$

From this, we express its *spectral decomposition* projection on this basis as

(5.244) $\wp = \lambda_+ P_\mathrm{u} + \lambda_- \overline{P}_\mathrm{u}, \quad$ and $\quad \overline{\wp} = \lambda_+ \overline{P}_\mathrm{u} + \lambda_- P_\mathrm{u},$

with the component coordinates for the paravector projections

(5.245) $\lambda_+ = (\lambda_0 + \lambda_\mathrm{u}) \quad$ and $\quad \lambda_- = (\lambda_0 - \lambda_\mathrm{u}) \quad \in \mathbb{R}.$

Conversely from these the coordinates from definition basis $\{1, \mathrm{u}\}$ for (5.242) is

(5.246) $\lambda_0 = ½(\lambda_+ + \lambda_-) \quad$ and $\quad \lambda_1 = ½(\lambda_+ - \lambda_-) \quad \in \mathbb{R}.$

$\{P_\mathrm{u}, \overline{P}_\mathrm{u}\}$ is orthogonal because $½(1 + \mathrm{u})½(1 - \mathrm{u}) = 0$, due to the first definition (5.56)-(5.59) :

(5.247) $P_\mathrm{u}\overline{P}_\mathrm{u} = P_\mathrm{u} \cdot \overline{P}_\mathrm{u} = P_\mathrm{u} \wedge \overline{P}_\mathrm{u} = 0.$

This condition also shows that the two basis paravectors are *mutual annihilating operations*, and we note their sum $P_\mathrm{u} + \overline{P}_\mathrm{u} = 1$ is the unit scalar, and their difference $P_\mathrm{u} - \overline{P}_\mathrm{u} = \mathrm{u}$ is the ***direction quality of first grade***.

[262] The name *paravector* is taken from William E. Baylis [37], [36], (First named by J. G. Maks, Ph.D. thesis, TU Delft, 1989.)
[263] Later below we look into 3 space with a paravector as $\wp = \lambda_0 + \mathbf{p} = \lambda_0 1 + \lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2 + \lambda_3\boldsymbol{\sigma}_3$, implying higher ***grades*** $\langle A\rangle_2, \langle A\rangle_3$, etc.
[264] To set the straight-line ***direction*** unit basis u of a paravector basis $\{1, \mathrm{u}\}$ in perspective we take the existence in a possible plane and multiply by a perpendicular unit $\mathrm{u}_\perp$ as in (5.208) and get $\{1, \mathrm{u}, \mathrm{u}_\perp, \boldsymbol{i} \equiv \mathrm{u}_\perp\mathrm{u}\}$. Then the paravector $\wp = \lambda_0 1 + \lambda_\mathrm{u}\mathrm{u} \longrightarrow$ goes to a multivector $\mathrm{u}_\perp\wp = \lambda_0\mathrm{u}_\perp + \lambda_\mathrm{u}\mathrm{u}_\perp\mathrm{u}$, that is nilpotent if $\lambda_\mathrm{u} = \pm\lambda_0$.