All the preceding pages 23-197 have first been written in Danish by the author, and primo 2017 translated into English by the same author.

The following chapters below are written directly in English. First chapter 6 is about natural space then afterwards followed by chapters 5.5-5.9 expanding the classical plane geometry, finalised by chapter 7 with Geometric Space-Time-Algebra for relations in natural Space.

5.5. Inherit Quantities of the Algebra for the Euclidean Geometric Plane Concept

We define the algebraic foundation of a geometric plane concept from two orthonormal basis vectors $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$. From these two 1-vectors, we define the bivector $\boldsymbol{i} = \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ as the outer product of two unit 1-vectors. $\boldsymbol{i}$ is called the unit pseudoscalar for that plane.
We form the 4-dimensional linear algebra for the Euclidean plane $\mathcal{G}_{2,0}(\mathbb{R}) = \mathcal{G}_2(\mathbb{R})$ by the generalised 2-multivector form as (5.161)

(5.197) $$M = \alpha + x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \qquad = \quad \alpha 1 + x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{i}$$

in the plane space spanned from the mixed 2^2-dimensional standard basis for the algebraic plane

(5.198) $$\{1, \ \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ \boldsymbol{i} := \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\}$$

This orthonormal mixed standard basis possesses a multiplication structure expressed in Table 5.1.

Table 5.1 Multiplication basis for the Euclidian plane algebra $\mathcal{G}_{2,0}(\mathbb{R})$ with the pseudoscalar unit $\boldsymbol{i} \equiv \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$

$\{1, \ \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ \boldsymbol{i}\}$

left \ right	1	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\boldsymbol{i}$
1	1	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\boldsymbol{i}$
$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_1$	1	$-\boldsymbol{i}$	$-\boldsymbol{\sigma}_2$
$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_2$	$\boldsymbol{i}$	1	$\boldsymbol{\sigma}_1$
$\boldsymbol{i}$	$\boldsymbol{i}$	$\boldsymbol{\sigma}_2$	$-\boldsymbol{\sigma}_1$	-1

The orthonormal basis

$\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_1 = 0,$

$\boldsymbol{\sigma}_k^2 = 1,$

$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = -\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1.$

$\{1, \ \boldsymbol{\sigma}_1, \ \boldsymbol{\sigma}_2, \ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\}$

left \ right	1	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$
1	1	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$
$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_1$	1	$-\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$	$-\boldsymbol{\sigma}_2$
$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$	1	$\boldsymbol{\sigma}_1$
$\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$-\boldsymbol{\sigma}_1$	-1

The possibility of products of two elements expands to all linear combinations of these in the form (5.197) and these 2-multivectors can be multiplied further *inside* the algebra $\mathcal{G}_{2,0}(\mathbb{R})$.

5.5.2. The Auto Product Square in the Euclidean plane

A simple product of 2-multivectors is the square of (5.197)

(5.199) $$M^2 = M(M) = MM = \quad (\alpha + x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1)^2 = \begin{aligned} &+(\alpha^2 + x_1^2 + x_2^2 - \beta^2)1 \\ &+ (2\alpha x_1)\boldsymbol{\sigma}_1 \\ &+ (2\alpha x_2)\boldsymbol{\sigma}_2 \\ &+ (2\alpha\beta)\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \end{aligned}$$

$M^2 = M(M)$, this multiplication is an auto operation of a multivector on itself.

5.5.3. The Nilpotent Operation

In some situations, the multiple operations with multivectors are vanishing and nihilate.
The simplest case is $M(M) = M^2 = 0$, hence we in $\mathcal{G}_{2,0}(\mathbb{R})$ by (5.199) express it as

(5.200) $$M^2 = (\alpha^2 + x_1^2 + x_2^2 - \beta^2) + (2\alpha x_1)\boldsymbol{\sigma}_1 + (2\alpha x_2)\boldsymbol{\sigma}_2 + (2\alpha\beta)\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \quad = 0.$$

For the nihilation, the scalar part must vanish, therefore an easy solution is to set

(5.201) $$\alpha = 0, \quad \text{and} \quad \beta^2 = x_1^2 + x_2^2 \quad \Rightarrow \quad \beta = \pm\sqrt{x_1^2 + x_2^2},$$

that works for $\forall x_k \in \mathbb{R}$, k=1,2. A nilpotent multivector in the Euclidean plane $\mathcal{G}_{2,0}(\mathbb{R})$ is then

(5.202) $$X = \ x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 \ \pm\sqrt{x_1^2 + x_2^2} \ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1.$$

Writing the Cartesian 1-vector $\mathbf{x} = x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2$, we can write the nilpotent multivector as

(5.203) $$X = \ \mathbf{x} \pm |\mathbf{x}|\boldsymbol{i} \qquad = \beta(\mathbf{u} \pm \boldsymbol{i}), \qquad \text{with} \ \ \mathbf{u} = \mathbf{x}/|\mathbf{x}|\,, \ \text{and} \quad \beta^2 = \mathbf{x}^2 > 0, \ \beta \in \mathbb{R}.$$

A test of this nihilation is due to that $\boldsymbol{i}\mathbf{x} + \mathbf{x}\boldsymbol{i} = 0$ express that $\mathbf{x} \parallel \boldsymbol{i}$ or $\mathbf{x} \wedge \boldsymbol{i} = \boldsymbol{i} \wedge \mathbf{x} = 0$.

(5.204) $$X(X) = X^2 = \quad \mathbf{x}^2 - |\mathbf{x}|^2 \ = 0, \qquad\qquad \text{this is equivalent to} \quad \big(\beta(\mathbf{u} \pm \boldsymbol{i})\big)^2 = 0.$$

Using the unit 1-vector ***direction*** $\mathbf{u} = \hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, in the plane ***direction*** $\boldsymbol{i} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ with β from (5.201) we write (5.203) for the Euclidean plane the *nilpotent multivector* as

(5.205) $$\beta(\mathbf{u} \pm \boldsymbol{i}), \qquad \text{where} \qquad \mathbf{u} \wedge \boldsymbol{i} = 0.$$