5.4.5. Rotation Inside the Same Plane *Direction*

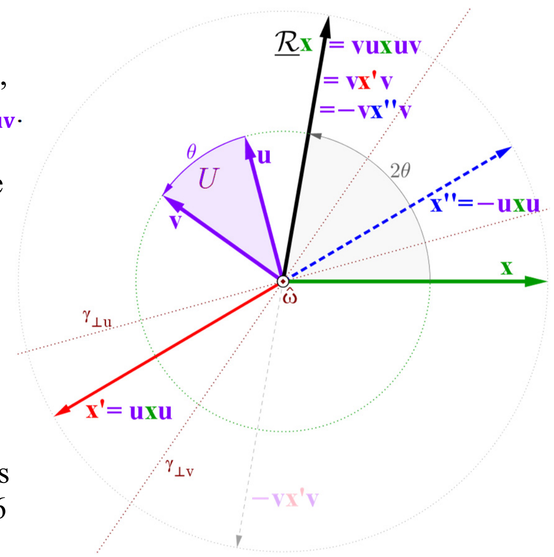

Figure 5.46 $\mathcal{R}$, Rotation in a plane. constituted of two reflections.

Two different normalized 1-vectors **u** and **v** where **v**≠**u**, |**u**|=|**v**|=1 form a bivector **v∧u**, which spans a plane $\lambda_{\mathbf{uv}}$. The product of these two also forms a 2-multi-vector U= **vu**, which works in the rotor plane $\lambda_{\mathbf{uv}}$, which is the plane of Figure 5.46 in which, there too are indicated two reflecting planes $\gamma_{\perp\mathbf{u}}\perp\mathbf{u}$ and $\gamma_{\perp\mathbf{v}}\perp\mathbf{v}$ as small-dotted lines for intuition. These are transversal normal planes to each of the 1-vectors **u** and **v**. These two plane objects $\gamma_{\perp\mathbf{u}}\perp\lambda_{\mathbf{uv}}$ and $\gamma_{\perp\mathbf{v}}\perp\lambda_{\mathbf{uv}}$ intersect each other in one straight line with the ***direction*** $\hat{\omega}\perp\lambda_{\mathbf{uv}}$, normal to the paper pointing towards the observer, who is seeded in a remote$^\infty$ origo and looks at the plane as an intuiting interpreter of the Figure 5.46 plane. We consider any arbitrary 1-vector **x**, in space $\mathfrak{G}$ whose projection on the $\lambda_{\mathbf{uv}}$ plane is showed Figure 5.46.
x can be reflected around **u**, then we get $\mathbf{x}'$ = **uxu**, or reflected along **u**, so that $\mathbf{x}''$= −**uxu** = $\mathcal{U}$**x** (dashed), is called an *irregular rotation*.
Hereafter $\mathbf{x}'$ is reflected around **v**, we get $\mathbf{vx}'\mathbf{v}$ or $\mathbf{x}''$ reflected along **v**, we achieve −**vx''v**=$\mathcal{V}$ **x''**, as another *irregular rotation*.
We compose these two reflections to one *regular rotation* $\mathcal{R} = \mathcal{V}\,\mathcal{U}$ (by double sandwiching)

(5.190) $\mathcal{R}\mathbf{x} = U\mathbf{x}U^\dagger = \mathbf{vuxuv} = -\mathbf{vx}''\mathbf{v} = -\mathbf{v}(-\mathbf{uxu})\mathbf{v}$ alternative $= \mathbf{vx}'\mathbf{v} = \mathbf{v}(\mathbf{uxu})\mathbf{v}$.

We see that the ambiguity of reflection through a 1-vector is eliminated by this doublet composition, where the multi-vector concept **vu** is the generator of the transformation.
We define a 2-multivector as an operator called a *rotor*

(5.191) $U = \mathbf{vu} = \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\wedge\mathbf{u} = e^{+\frac{1}{2}\boldsymbol{\theta}} = e^{+\boldsymbol{i}\theta}$

(5.192) $U^\dagger = \mathbf{uv} = \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\wedge\mathbf{u} = e^{-\frac{1}{2}\boldsymbol{\theta}} = e^{-\boldsymbol{i}\theta}$

Here we introduce the bivector $\frac{1}{2}\boldsymbol{\theta} = \boldsymbol{i}\theta$ which represents the rotor area with ***direction***, as an argument for this 2-multivector exponential function $e^{\pm\frac{1}{2}\boldsymbol{\theta}}$.
The unit of plane ***direction*** is $\boldsymbol{i}$=$\hat{\boldsymbol{\theta}}$, refer to (5.8), (5.90).
Hence, the regular rotation as a linear transformation is

(5.193) $\mathcal{R}\mathbf{x} = U\mathbf{x}U^\dagger = e^{\frac{1}{2}\boldsymbol{\theta}}\mathbf{x}\,e^{-\frac{1}{2}\boldsymbol{\theta}} = e^{\boldsymbol{\theta}}\mathbf{x} = e^{\boldsymbol{i}\theta}\mathbf{x}e^{-\boldsymbol{i}\theta} = e^{\boldsymbol{i}2\theta}\mathbf{x}$

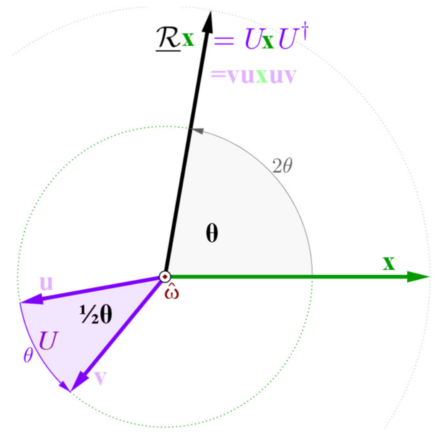

Figure 5.47 The *rotation* is a linear angular ***pqg***-2 *transformation* $U = e^{+\boldsymbol{i}\theta}$.

The unitary rotor 2-multivector is rotational invariant in its own plane. This means, its ***pqg***-1 ***directions*** by the individual 1-vectors **u** and **v** lose their specific meaning, and instead their relative ***direction*** area ½**θ**, which is the ***pqg***-2 ***direction*** of that rotation area.
The 1-rotor is

(5.194) $U_\theta = e^{+\frac{1}{2}\boldsymbol{\theta}} = e^{+\boldsymbol{i}\theta}$,

and the *regular rotation* is $\mathcal{R}\mathbf{x} = U\mathbf{x}U^\dagger$ in Figure 5.47, (by rotor sandwiching)
If the start 1-vector **x** is in the rotor plane, we can choose U= **uv** = **ux**/|**x**| the resulting 1-vector $U^2\mathbf{x} = e^{\boldsymbol{i}2\theta}\mathbf{x}$ is in that same plane reduced to a reflection (5.170) $\mathbf{x}'$=**uxu** displayed in Figure 5.42 where **v** ∥ **x** is omitted. –

The general formulation (5.193) $\mathcal{R}\mathbf{x} = U\mathbf{x}U^\dagger$ apply to all vectors in space $\mathfrak{G}$ outside the rotor U plane ***direction***, see more below § 6.3.3, Figure 6.12.

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This form $\mathcal{R}\mathbf{x} = U\mathbf{x}U^\dagger$ is often called the canonical form for any orthogonal transformation $\mathcal{R}$.
This is the ***pqg***-2 essence of rotation, the ***primary quality of the second grade*** for all space $\mathfrak{G}$ of physics. A rotation plane ***direction*** is the same subject all over space $\mathfrak{G}$ (translation invariance).
This ***pqg***-2 symmetry is an a priori foundation for all space $\mathfrak{G}$ in physics.

5.4.5.2. The Half-Angle Rotor of an Euler Rotation

We see that when we have a 1-rotor (5.194) $U_\theta = e^{+\boldsymbol{i}\theta}$ the full regular rotation has an argument with the double angle $\mathcal{R}\mathbf{x} = U\mathbf{x}U^\dagger = U^2\mathbf{x} = e^{\boldsymbol{i}2\theta}\mathbf{x}$. Therefore, we almost always write a 1-rotor with a half angle $\frac{1}{2}\varphi$ argument causing the full regular rotation with the full angle φ in this

(5.195) $U_\varphi = e^{+\boldsymbol{i}\frac{1}{2}\varphi} \;\Rightarrow\; U_\varphi^2 = e^{+\boldsymbol{i}\varphi} \;\Rightarrow\; \mathcal{R}\mathbf{x} = U_\varphi\mathbf{x}U_\varphi^\dagger = e^{+\boldsymbol{i}\frac{1}{2}\varphi}\mathbf{x}e^{-\boldsymbol{i}\frac{1}{2}\varphi} = U_\varphi^2\mathbf{x} = e^{+\boldsymbol{i}\varphi}\mathbf{x}$.

This we call an Euler rotation, and if the rotor $e^{\pm\boldsymbol{i}\frac{1}{2}\varphi}$ is in one fixed plane $\boldsymbol{i}$ we call it a 1-rotor.
The purpose of half angles $\frac{1}{2}\varphi$ is clear when it comes to Euler rotations in several planes .

5.4.5.3. The Idea of an Active Rotation

As we know from the treatment in chapters 1, 2 and 3, to experience physics something happens. There always is a development, and the measure for this is a development parameter t given from a rotating circle oscillator with frequency energy ω. We now see that this circle oscillation is a rotation along a Euclidean plane. We simply write this unitary as a plane rotation operator

(5.196) $e^{\pm\boldsymbol{i}\omega t} = e^{\pm\boldsymbol{i}\varphi} = U_\varphi^2$

where $\boldsymbol{i}$ is the unit bivector ***direction*** of the Euclidean rotation plane, and $\omega t = \varphi$ is the active ***quantum mechanical*** phase angle of the oscillating ***entity***, which describes the development.[259]
A 1-rotor for an oscillation is often just written $U_\varphi = e^{\pm\boldsymbol{i}\frac{1}{2}\varphi} = e^{\pm\boldsymbol{i}\frac{1}{2}\omega t}$.

5.4.5.4. The Invariant Direction of a Rotor

The rotational invariance of the rotor object U= **vu**= **v**·**u**+**v∧u** is illustrated by the difference from Figure 5.46 to Figure 5.47 in accordance with the idea in § 5.2.7.4 and Figure 5.23.
By normal duality, we can characterise the rotation plane $\lambda_U = \lambda_{\mathbf{uv}} = \lambda_{\mathbf{u}\wedge\mathbf{v}} = \lambda_{\boldsymbol{\theta}} = \lambda_{\boldsymbol{i}} = \lambda_{\perp\hat{\omega}}$ by a new 1-vector $\hat{\omega}$, $|\hat{\omega}|$=1 , which is the normal vector to the rotation plane $\hat{\omega}\perp\lambda_U$.
This ***pqg***-1-vector subject is perpendicular to the 'paper' plane in Figure 5.46 and Figure 5.47 and indicates the ***direction***, that light has to face the observer from the rotation plane.
– Imagine that you look perpendicular to the figure plane surface right on.[260]

5.4.5.5. The Duality of *Direction*

The light you as the reader of this book receive and see has two characteristics:

- You receive it as a particle expressed as a momentum ***pqg***-1-vector $\frac{\hbar}{c}\;\omega\hat{\omega}$, or
- You receive it as a transversal plane wave expressed through a unitary development 1-rotor $U_\omega(t) = e^{\pm\boldsymbol{i}\theta} = e^{\pm\boldsymbol{i}\omega t} \sim$ **be** = **b**·**e** + **b∧e**. We presume orthogonality **b**·**e**=0. Then the field bivector **F**= **b∧e** ~∥ $\boldsymbol{i}$ gives the plane ***pqg***-2 ***direction*** of the wavefront.

These two are mutual dual following the complementarity principle between particle ***pqg***-1 ***direction*** and the plane wavefront ***pqg***-2 ***direction***, where the plane rotor is a combination of a scalar ***pqg***-0 quality without ***direction*** and the bivector plane ***direction pqg***-2 ***quality*** as an even geometric algebra.
The idea of a particle picture of a physical ***entity*** will be concerned with the odd part of Geometric Algebra. More of this below in the following chapters.

[259] We will below in section 5.7.5 interpret the development as a ***direction*** unit called γ_0 as a 1-vector in Space-Time-Algebra.
[260] You cannot see $\hat{\omega}$ in perspective, which causes the viewing angle loses its meaning. (An abstraction automatic don in your brain.) Later in Section 5.7.4, this will give analytical meaning by the null direction of a Lorentz rotation. The duality concept problem will be further analysed in chapter 6.

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