5.4.2.2. Reflection Along a Geometric 1-vector

Multiplying the reflected 1-vector $\mathbf{x}' = \mathbf{uxu}$ with the factor -1, we get the reflection image $\mathbf{x}'' = -\mathbf{uxu} = -\mathbf{x}'$, called the reflection of a 1-vector $\mathbf{x}$ along the given 1-vector $\mathbf{u}$.
We see the mirror image $\mathbf{x}''$ through a line $\ell_{\perp\mathbf{u}}$ or a plane $\gamma_{\perp\mathbf{u}}$ perpendicular⊥ (transversal) to the given 1-vector $\mathbf{u}$.
The bivector plane $\gamma_{\mathbf{u}\wedge\mathbf{x}}$ spanned by $\mathbf{u}\wedge\mathbf{x}$ is the paper plane of Figure 5.43, in which all the 1-vectors $\mathbf{u}, \mathbf{x}, \mathbf{x}', \mathbf{x}''$ exist as objects. We introduce another plane subject $\gamma_{\perp\mathbf{u}}$, called a normal plane to the 1-vector $\mathbf{u}$, a so-called reflecting plane. The reflection of $\mathbf{x}$ around $\mathbf{u}$ on this reflecting plane $\gamma_{\perp\mathbf{u}}$ is then $\mathbf{x}' = \mathbf{uxu}$, and through the plane γ we see the mirror image

(5.177) $\quad \mathbf{x}'' = -\,\mathbf{uxu} \quad = -\mathbf{x}'$

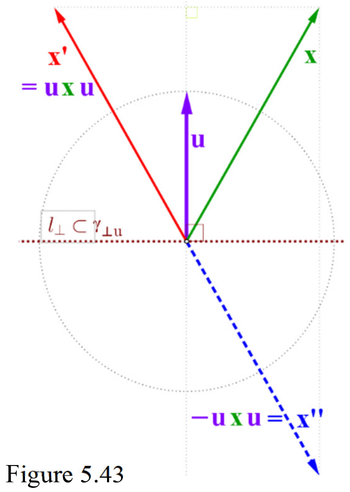

Figure 5.43
Reflection in a normal plane ⊥ to $\mathbf{u}$ is equivalent to a reflection along $\mathbf{u}$.

Conversely, given a plane γ surface object in space $\mathfrak{G}$, this can be assigned a normal 1-vector $\mathbf{u}$ perpendicular $\mathbf{u}\perp\gamma$ to the plane and normalized $|\mathbf{u}|=1$, thus generating the reflections $\pm\mathbf{uxu}$ of any arbitrary 1-vector $\mathbf{x}$.
Note that this simplest algebraic form for a ***pqg*-2 *quality*** reflection has two possible orientations of its outcome $\pm\mathbf{uxu}$.

5.4.2.3. Reflection Through a Non-normalized 1-vector

The two opposite-orientated reflection formulas $\pm\mathbf{uxu}$ both ‘sandwiching’ the arbitrary 1-vector $\mathbf{x}$ between the given normalized reflection 1-vector $\mathbf{u}$. If the given 1-vector is not normalized $\mathbf{a} = |\mathbf{a}|\hat{\mathbf{a}} = |\mathbf{a}|\mathbf{u}$, the reflection formulas are simply written

(5.178) $\quad \pm\mathbf{a}^{-1}\mathbf{x}\mathbf{a} = \ \pm\mathbf{a}\mathbf{x}\mathbf{a}^{-1} = \ \pm\hat{\mathbf{a}}\mathbf{x}\hat{\mathbf{a}} \sim \ \pm\mathbf{uxu}.$ (for + see **a** Figure 5.44.)

5.4.3. The Projection Operator From one 1-vector to Another 1-vector

Having any given 1-vector $\mathbf{a}$ with a unit ***direction*** as $\mathbf{u} = \hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$, we will create the projection of each arbitrary 1-vector ***direction*** $\mathbf{x}$ in space on this 1-vector ***direction*** $\hat{\mathbf{a}}$ of $\mathbf{a}$, just as (5.173).
We use the inverse of $\mathbf{a}$ (5.69) $\mathbf{a}^{-1} = \mathbf{a}/a^2 \Rightarrow \mathbf{a}\mathbf{a}^{-1}=1$. The product of these two 1-vectors is

(5.179) $\quad \mathbf{xa} = \mathbf{x}\cdot\mathbf{a} + \mathbf{x}\wedge\mathbf{a} \quad \Longrightarrow \quad \mathbf{x} = \mathbf{x}\mathbf{a}\mathbf{a}^{-1} = (\mathbf{x}\cdot\mathbf{a} + \mathbf{x}\wedge\mathbf{a})\mathbf{a}^{-1} = \ (\mathbf{x}\cdot\mathbf{a})\mathbf{a}^{-1} + (\mathbf{x}\wedge\mathbf{a})\mathbf{a}^{-1} = \ \mathbf{x}_{\parallel\mathbf{a}} + \mathbf{x}_{\perp\mathbf{a}},$

where we have divided $\mathbf{a}$ out again and achieved the parallel component (see §5.2.2.3 scalar product)

(5.180) $\quad \mathbf{x}_{\parallel\mathbf{a}} \coloneqq \mathbf{a}^{-1}\mathbf{a}\cdot\mathbf{x} \quad \coloneqq \ \mathbf{a}^{-1}(\mathbf{a}\cdot\mathbf{x}) = \ (\mathbf{x}\cdot\mathbf{a})\mathbf{a}^{-1} \quad = \ \mathbf{x} - (\mathbf{x}\wedge\mathbf{a})\mathbf{a}^{-1}.$

We note the parallel symmetric and the orthogonal antisymmetric components |

(5.181) $\quad \mathbf{x}_{\parallel\mathbf{a}}\mathbf{a} \ = \ \mathbf{x}\cdot\mathbf{a} \ = ½(\mathbf{xa} + \mathbf{ax}) = \mathbf{a}\cdot\mathbf{x} \ = \ ½(\mathbf{ax} + \mathbf{xa}) = \mathbf{a}\mathbf{x}_{\parallel\mathbf{a}},$

(5.182) $\quad \mathbf{x}_{\perp\mathbf{a}}\mathbf{a} = \ \mathbf{x}\wedge\mathbf{a} = ½(\mathbf{xa} - \mathbf{ax}) \ = -\mathbf{a}\wedge\mathbf{x} = \ -\mathbf{a}\mathbf{x}_{\perp\mathbf{a}}.$

We left multiply with $\mathbf{a}^{-1}$ in the last form of the symmetry in (5.181)

(5.183) $\quad \mathbf{x}_{\parallel\mathbf{a}} = \mathbf{a}^{-1}\mathbf{a}\mathbf{x}_{\parallel\mathbf{a}} = ½\mathbf{a}^{-1}(\mathbf{ax} + \mathbf{xa}) = ½(\mathbf{x} + \mathbf{a}^{-1}\mathbf{xa}).$

This last is illustrated as objects for intuition in Figure 5.44.
Using (5.180) we have now introduced a *projection operator*

(5.184) $\quad \boxed{P_{\mathbf{a}}\mathbf{x} \coloneqq \mathbf{a}^{-1}\mathbf{a}\cdot\mathbf{x} \ = (\mathbf{x}\cdot\mathbf{a})\mathbf{a}^{-1} \ = \ P_{\mathbf{a}}(\mathbf{x}) = \ ½(\mathbf{x} + \mathbf{a}^{-1}\mathbf{xa})},$

that is a *linear transformation* inside a Euclidean space [10]p.253.
Such projection is an *idempotent* linear transformation operation

(5.185) $\quad P_{\mathbf{a}}^{\,2} = P_{\mathbf{a}} \quad \Leftrightarrow \quad P_{\mathbf{a}}\big(P_{\mathbf{a}}(\mathbf{x})\big) = P_{\mathbf{a}}(\mathbf{x}),$

that says, one projection has an impact, on further projections on $\mathbf{a}$ have no impact.

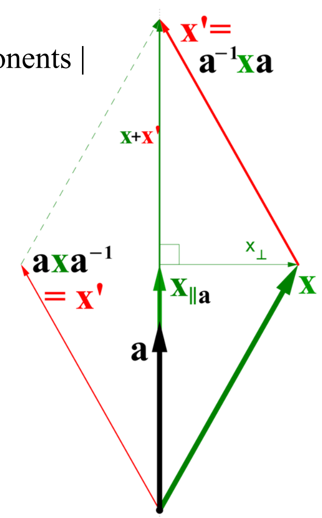

Figure 5.44. The projection of $\mathbf{x}$ along $\mathbf{a}$ is intuited as $P_{\mathbf{a}}(\mathbf{x}) = \mathbf{x}_{\parallel\mathbf{a}} = ½(\mathbf{x} + \mathbf{a}^{-1}\mathbf{xa}) = ½(\mathbf{x}+\mathbf{x}')$.

5.4.4. Reflection in a Plane Surface as a Physical Process

Observing a *particle* $\Psi_{\mathbf{p}}$ ***direction quantity pqg***-1-vector $\mathbf{p}$ (such as light), which we consider as reflected in another physical ***entity*** Ψ_γ, that we assign a mirroring plane γ or just a mirroring line $\ell \subset \gamma$ such that we assume that the incident particle is given $\mathbf{p}_{in} = -\mathbf{upu}$, as shown in Figure 5.45, and assuming $\mathbf{u}\perp\gamma \Rightarrow \mathbf{u}\perp\ell$ is the normal 1-vector $\mathbf{u}$ that perpendicular defines the ***direction*** of both ℓ and γ for the surface of the reflecting ***entity*** Ψ_γ.
We have from the quantum concept that $|\mathbf{p}_{in}| = |\mathbf{p}|$.
We look at the change of the particle 1-vector[256]

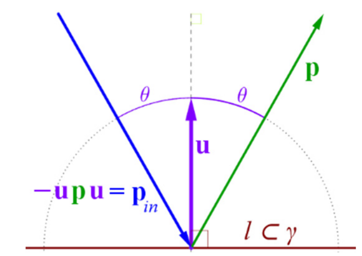

Figure 5.45 Particle reflection inside the plane $\gamma_{\mathbf{u}\wedge\mathbf{p}} = \gamma_{\mathbf{up}}$ by the normal 1-vector $\mathbf{u}$ to a reflecting line ℓ in the plane $\gamma\perp\mathbf{u}$ of a reflecting surface Ψ_γ.

(5.186) $\quad \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_{in}\,, \quad$ which has the ***direction*** $\quad \mathbf{u} = \frac{\Delta\mathbf{p}}{|\Delta\mathbf{p}|}$.

The simplest linear transformation that can be formed along a finite 1-vector $\Delta\mathbf{p} = |\Delta\mathbf{p}|\mathbf{u}$, $(\Delta\mathbf{p}\neq\mathbf{0})$ or just along its normalized 1-vector $\mathbf{u}$ is written as

(5.187) $\quad \underline{\mathcal{U}}\mathbf{p} = \ \mathbf{p}_{\text{in}} = \ -\mathbf{upu} = \ -\Delta\mathbf{p}^{-1}\mathbf{p}\Delta\mathbf{p}$

From (5.170) we have $\mathbf{p}_{in} = -\mathbf{upu} = -U\mathbf{p}U^{\dagger} = -UU\mathbf{p} = -U^2\mathbf{p} = \underline{\mathcal{U}}\mathbf{p}$, where the 1-rotor $U_\theta = \mathbf{u}\hat{\mathbf{p}} = \frac{\mathbf{up}}{|\mathbf{p}|} = e^{\mathbf{i}\theta}$ works twice U_θ^2 in the same plane.
The reflection transformation is then expressed as its cause $\mathbf{p}_{\text{in}} =$

(5.188) $\quad \underline{\mathcal{U}}\mathbf{p} = -U_\theta^{\,2}\mathbf{p} = -\,e^{\mathbf{i}2\theta}\mathbf{p},$

which is called an *irregular rotation*.
The incident angle of reflection is equal to the angle of departure

(5.189) $\quad \theta = \theta_{out} = \theta_{in},$

and the total angle is 2θ.
This entire process in this physics takes place through the plane of the same reflection plane $\gamma_{\mathbf{up}}$, spanned by the 2-multi-vector $\mathbf{up}$ or just[257] the bivector $\mathbf{u}\wedge\mathbf{p}$.
All the 1-vector objects $\mathbf{p}$, $\mathbf{p}_{in}$, $\Delta\mathbf{p}$, and $\mathbf{u}\perp\ell$ for the intuition[258] exist in that foundation plane $\gamma_{\mathbf{up}}$ of Figure 5.45. We may say that reflection takes place along the 1-vector $\mathbf{u}$.
For reflection, we can introduce a mirror plane γ with $\mathbf{u}$ as the normal 1-vector ***direction***, as well as the mirroring line $\ell \subset \gamma$. This plane must have its cause in the intended physical ***entity*** Ψ_γ.
That reflecting mirror plane γ for the intuition is perpendicular to the reflection plane $\gamma_{\mathbf{up}}$, $\gamma\perp\gamma_{\mathbf{up}}$, and therefore outside the paper plane of Figure 5.45. This γ is called the normal plane of the 1-vector $\mathbf{u}$ ***direction***.
We would often prefer to call a plane $\gamma\perp\mathbf{u}$ for the transversal plane of a 1-vector $\mathbf{u}$ ***direction***.
But by this, we have introduced an additional external dimension to the Figure 5.45 plane of the reflection objects $\mathbf{u}$ etc. This ***pqg***-2 concept of a transversal plane is just dual exterior transversal to a ***pqg***-1-vector, which will be treated later below in section 6.2.4.

Throughout this chapter 5.4, we have seen 1-vectors, bivectors and 2-multi-vectors as existing in the same plane, namely the bivector-co-plane $\gamma_{\mathbf{u}\wedge\mathbf{p}} = \gamma_{\mathbf{up}}$. Therefore, we have intuited the reflection concept purely in this foundation plane of Figure 5.45. Then the reflecting plane is seen purely as a line $\ell \subset \gamma$ for our intuition.

[256] We ignore the recoil changes $\Delta\mathbf{p}_\gamma$ for the physical ***entity*** Ψ_γ, $\Delta\mathbf{p}_\gamma + \Delta\mathbf{p} = 0$, since the reference frame for the system is Ψ_γ.
[257] Since the ***pqg***-0 scalar $\mathbf{u}\cdot\mathbf{p}$ does not affect the ***pqg***-2 ***direction*** of the plane.
[258] The knowledge $2\mathbf{p}_\perp = \mathbf{p} + \mathbf{p}_{in}$, $\mathbf{u}\cdot\mathbf{p}_\perp = 0$ is irrelevant to this interpretation of a reflection by a geometric multiplication algebra.