This general 2-multi-vector is constructed from three 1-vectors

- An arbitrary *norm* and start ***direction*** 1-vector $\mathbf{u}$, with $|\mathbf{u}| = 1$ unit.
- An arbitrary related *radius* ***direction*** 1-vector $\mathbf{r}$, with $|\mathbf{r}| = \rho$ dilation.
- An arbitrary *translation* ***direction*** 1-vector $\mathbf{t}$, with $\mathbf{t} \parallel (\mathbf{r}\wedge\mathbf{u})$ or $\mathbf{t} \nparallel (\mathbf{r}\wedge\mathbf{u})$.

$\mathbf{r}\cdot\mathbf{u} = \alpha$ is the geometric ***pqg***-0 scalar, $\langle A\rangle_0$
$\mathbf{r}\wedge\mathbf{u} = \beta\boldsymbol{i}$ is the geometric plane ***direction pqg***-2 *bivector*, $\langle A\rangle_2$
$\mathbf{ru}$ is the geometric ***spinor*** in that plane ***direction***, $\langle A\rangle_0 + \langle A\rangle_2$
$\mathbf{t} \parallel (\mathbf{r}\wedge\mathbf{u})$ is a *translation* inside that same *object plane* ***direction***, $\langle A\rangle_1$
$\mathbf{t} \nparallel (\mathbf{r}\wedge\mathbf{u})$ is a *translation* of that object plane through space as that same plane *subject* ***direction pqg***-2, which is *translation invariant* through all space $\mathfrak{G}$!
One specified plane subject ***direction*** is the same all over space $\mathfrak{G}$ in physics.[254]

A traditional plane object can be defined by two orthonormal basis vectors $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\} \parallel (\mathbf{r}\wedge\mathbf{u})$ [255] implying the bivector concept $\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$. In that plane, we express every arbitrary 1-vector

(5.160) $\langle A\rangle_1 \sim \mathbf{x} = x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2.$ Special a translation $\mathbf{t} = t_1\boldsymbol{\sigma}_1 + t_2\boldsymbol{\sigma}_2$.

as a vector element in an ordinary vector space $\mathbf{x} \in (V_2, \mathbb{R}) \sim \mathbb{R}^2_1$.
In the geometric algebra $\mathcal{G}_2(\mathbb{R}) = \mathcal{G}(V_2, \mathbb{R})$, we express it as (5.159) for the 2-multi-vector concept

(5.161) $$A = \alpha + \underbrace{x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2} + \beta\boldsymbol{i} \quad \in \mathcal{G}_2(\mathbb{R})$$

(5.162) $$A = \langle A\rangle_0 \; + \; \langle A\rangle_1 \; + \langle A\rangle_2$$

We as David Hestenes [10] p.53 split the plane geometric algebra into two parts $\mathcal{G}_2 = \mathcal{G}_2^- + \mathcal{G}_2^+$, where the odd algebra $\mathcal{G}_2^-$ is the ordinary 2-dimensional vector space $(V_2, \mathbb{R}) \sim \mathbb{R}^1_1 \oplus \mathbb{R}^1_2 \leftrightarrow \mathbb{R}^2_{1,2}$ for 1-vectors $\mathbf{x} = \langle A\rangle_1 \in \mathcal{G}_2^-$, and the even algebra $\mathcal{G}_2^+$ is the 2-dimensional linear space of 1-spinors

(5.163) $$\langle A\rangle_0 + \langle A\rangle_2 = \alpha + \beta\boldsymbol{i} = \mathbf{r}\cdot\mathbf{u} + \mathbf{r}\wedge\mathbf{u} = \mathbf{ru} = \rho U_\theta = \rho e^{\boldsymbol{i}\theta} \quad \in \mathcal{G}_2^+.$$

The object foundation of this even algebra $\mathcal{G}_2^+$ is the fundamental scalar count $1 \in \mathbb{R}$ and the orientated unit area $\boldsymbol{i}$ as the basis set $\{1, \boldsymbol{i}\}$. Together these two algebras $\mathcal{G}_2 = \mathcal{G}_2^- + \mathcal{G}_2^+$ constitute a 4-dimensional linear space over the multi-vector basis

(5.164) $$\{1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{i} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\},$$

with the odd coordinates x_1, x_2 and the even coordinates α, β as in (5.161).
The entire 2-multi-vector A has the potential to characterise the plane ***quality*** as well as a ***quantity*** for some physical ***entity***. – For the Euclidean plane algebra, we use the nomenclature $\mathcal{G}_{2,0}(\mathbb{R}) \leftarrow \mathcal{G}_2$.

5.3.8. The Magnitude of a Multivector

The generalised multivector A can be dissolved in its independent ***grade*** components $\langle A\rangle_r$

(5.165) $$A = \sum_{r=0}^{n} \langle A\rangle_r = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \cdots + \langle A\rangle_n \quad \in \mathcal{G}_n = \mathcal{G}_n(\mathbb{R}).$$

Note that each $\langle A\rangle_r$ of ***grade-r*** for $n \geq r > 2$ is an external extension to the plane concept. The magnitude, also called the *modulus* of a multi-vector is generally defined [10]p.46 and [13] p.13:

(5.166) $$|A| = \sqrt{\langle A^\dagger A\rangle_0}, \quad \text{so that} \quad |A|^2 = \left\langle A^\dagger A\right\rangle_0 = \sum_{r=0}^{n} A_r^\dagger A_r = \sum_{r=0}^{n} |\langle A\rangle_r|^2 \quad \geq 0, \text{ written out}$$

(5.167) $$|A|^2 = \left\langle A^\dagger A\right\rangle_0 = |\langle A\rangle_0|^2 + |\langle A\rangle_1|^2 + |\langle A\rangle_2|^2 + |\langle A\rangle_3|^2 + \cdots \quad \geq 0.$$

≥ 0 is a simplified pure mathematic assumption. – More generally about multi-vectors for physics later below.

[254] The ***direction*** of a plane object we define as $\boldsymbol{i} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ from the two objects 1-vectors $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$. For describing the translation through space $\mathfrak{G}$ outside the plane object we need a third 1-vector $\boldsymbol{\sigma}_3 \perp (\boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1)$ to make an orthonormal basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ for the 1-vector combination $\mathbf{t} = t_1\boldsymbol{\sigma}_1 + t_2\boldsymbol{\sigma}_2 + t_3\boldsymbol{\sigma}_3 \in (V_3, \mathbb{R}) \sim \mathbb{R}^3_{1,2,3}$ representing the translation. $\mathbf{t} \nparallel (\boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1) \Leftrightarrow t_3 \neq 0$. In plane $t_3 = 0$.

[255] In that plane we have $\langle A\rangle_2 \sim (\mathbf{r}\wedge\mathbf{u}) \parallel (\boldsymbol{i} = \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1)$, then we can express $\mathbf{u} = u_1\boldsymbol{\sigma}_1 + u_2\boldsymbol{\sigma}_2$ and $\mathbf{r} = r_1\boldsymbol{\sigma}_1 + r_2\boldsymbol{\sigma}_2$ as $\langle A\rangle_1$, choice $\boldsymbol{\sigma}_1 = \mathbf{u}$.

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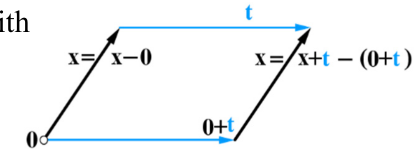
5.4. Transformation of Geometric 1-vectors in the Euclidean plane

5.4.1. Parallel Translation of a Vector

As we described earlier in § 4.4.2.13, the *points* in space $\mathfrak{G}$ are translations invariant.
We also decide as a priori that scalar magnitudes associated with the *locus situs* of geometric points is translations invariant.
We have also found that 1-vectors are translation invariant, since the parallel translation concept is based on the substance of the 1-vector concept. We refer to the previous Figure 4.6 (which is shown here).



The same will inherently also apply to multi-vectors, as in their idea they are constructed from 1-vectors of the Geometric Algebra of products and additions. The now well-known simple 2-multi-vector 1-rotor $\mathbf{vu} = \mathbf{u}_2\mathbf{u}_1 = U_\theta \coloneqq e^{\boldsymbol{i}\theta}$ from (5.83) is translation invariant.

5.4.2. Reflections

5.4.2.1. Reflection in a Geometric 1-vector

Given a 1-vector $\mathbf{u}$ in $\mathfrak{G}$ space. We choose $\mathbf{u}$ as norm $|\mathbf{u}| = 1$.
We choose any other object 1-vector $\mathbf{x}$ in $\mathfrak{G}$ space.
These two 1-vector form a plane spanned by the bivector $\mathbf{u}\wedge\mathbf{x}$
This plane is the foundation object for Figure 5.41.
We make a normalized 1-vector ***direction*** $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ for $\mathbf{x} = |\mathbf{x}|\hat{\mathbf{x}}$.
For understanding, we have $\hat{\mathbf{x}}\hat{\mathbf{x}} = \hat{\mathbf{x}}^2 = 1$ and $\mathbf{xx} = \mathbf{x}^2 = |\mathbf{x}||\mathbf{x}| \in \mathbb{R}_+$.
The 2-vector $\mathbf{u}\hat{\mathbf{x}}$ forms a rotor $U = \mathbf{u}\hat{\mathbf{x}} = \mathbf{ux}/|\mathbf{x}|$ in plane $\mathbf{u}\wedge\mathbf{x}$.
The 1-rotor was first defined in formulas (5.83) and (5.84) above

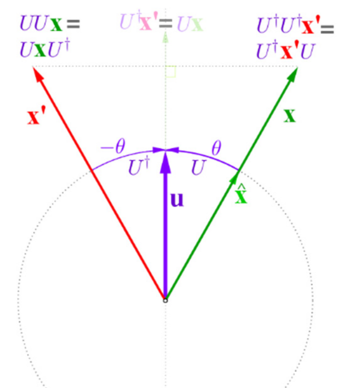


Figure 5.41 The plane $\mathbf{u}\wedge\mathbf{x}$.

Note the unit circle for the foundation plane idea.

(5.168) $$U_\theta = \mathbf{u}_2\mathbf{u}_1 = e^{\boldsymbol{i}\theta}, \quad \text{and} \quad U_\theta^\dagger = \mathbf{u}_1\mathbf{u}_2 = e^{-\boldsymbol{i}\theta}, \quad \text{with } \sphericalangle(\mathbf{u}_1, \mathbf{u}_2) = \theta.$$

When the rotor U acts on $\mathbf{x}$, we get $U\mathbf{x} = \mathbf{u}\hat{\mathbf{x}}\mathbf{x} = \mathbf{uxx}/|\mathbf{x}| = |\mathbf{x}|\mathbf{u}$.
We are searching the symmetrically reflected 1-vector $\mathbf{x}'$ to $\mathbf{x}$ around $\mathbf{u}$, still in the $\mathbf{u}\wedge\mathbf{x}$ plane.
The reverse rotor $U^\dagger = \hat{\mathbf{x}}\mathbf{u}$ (5.84) acts on $\mathbf{x}'$ and must symmetrically give

(5.169) $$U^\dagger\mathbf{x}' = |\mathbf{x}'|\mathbf{u} \coloneqq |\mathbf{x}|\mathbf{u} = U\mathbf{x}, \quad \text{as } |\mathbf{x}'| \coloneqq |\mathbf{x}|$$ (dimmed in the top middle of Figure 5.41)

Let U act on this 1-vector $U\mathbf{x}$ once again $U^2\mathbf{x} = UU\mathbf{x} = \frac{\mathbf{uxuxx}}{|\mathbf{x}||\mathbf{x}|} = \mathbf{uxu} = \frac{\mathbf{uxxu}}{|\mathbf{x}||\mathbf{x}|} = U\mathbf{x}U^\dagger$

As a result, we have the rule of reflection of 1-vectors $\mathbf{x}$ through (around) a given 1-vector $\mathbf{u}$

(5.170) $$\mathbf{x}' = \mathbf{uxu} = U\mathbf{x}U^\dagger, \quad \text{and reverse}$$

(5.171) $$\mathbf{x} = \mathbf{ux}'\mathbf{u} = U^\dagger\mathbf{x}'U, \quad \text{as shown in Figure 5.42.}$$

This is the fundamental formulation of reflection in $\mathbf{u}$ inside this plane of Figure 5.41 and Figure 5.42. (around $\mathbf{u}$)
We can divide any 1-vector into components along $\mathbf{u}$, like

(5.172) $$\mathbf{x}_\parallel = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}, \quad \text{and transverse} \quad \mathbf{x}_\perp = (\mathbf{x}\wedge\mathbf{u})\mathbf{u}, \quad \text{as}$$

(5.173) $$\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp \quad \Leftrightarrow \quad \mathbf{x}' = \mathbf{uxu} = \mathbf{x}_\parallel - \mathbf{x}_\perp, \quad \text{showed in Figure 5.42}$$

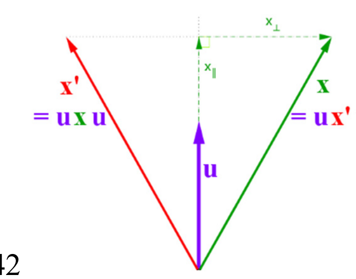


Figure 5.42 Reflection around a 1-vector $\mathbf{u}$. (through $\mathbf{u}$ in the plane).

This is because we from (5.58)-(5.59) can write a 2-multivector as

(5.174) $$\mathbf{ux} = \mathbf{u}\cdot\mathbf{x} + \mathbf{u}\wedge\mathbf{x} \quad \Leftrightarrow \quad \mathbf{xu} = \mathbf{u}\cdot\mathbf{x} - \mathbf{u}\wedge\mathbf{x} = \mathbf{x}\cdot\mathbf{u} + \mathbf{x}\wedge\mathbf{u},$$

and as we have

(5.175) $$\mathbf{u}\cdot\mathbf{x}_\perp = \mathbf{x}_\perp\cdot\mathbf{u} = 0, \quad \text{and} \quad \mathbf{u}\wedge\mathbf{x}_\parallel = \mathbf{x}_\parallel\wedge\mathbf{u} = 0; \quad \text{and} \quad \mathbf{u}\cdot\mathbf{x}_\parallel = \mathbf{x}_\parallel\cdot\mathbf{u}, \quad \text{and} \quad \mathbf{u}\wedge\mathbf{x}_\perp = -\mathbf{x}_\perp\wedge\mathbf{u},$$

Hence (5.173)

(5.176) $$\mathbf{ux} = \mathbf{u}\cdot\mathbf{x}_\parallel + \mathbf{u}\wedge\mathbf{x}_\perp \Leftrightarrow \mathbf{xu} = \mathbf{u}\cdot\mathbf{x}_\parallel - \mathbf{u}\wedge\mathbf{x}_\perp \quad \Leftrightarrow \quad \mathbf{x}' = \mathbf{uxu} = \mathbf{uu}\cdot\mathbf{x}_\parallel - \mathbf{uu}\wedge\mathbf{x}_\perp = \mathbf{x}_\parallel - \mathbf{x}_\perp.$$