5.3.4. The *Qualities* of the Geometric Algebra of the Plane

5.3.4.1. The Rotation Symmetrical Plane Concept

The idea that a bivector is a rotational invariant in its own plane makes us guess that there exist physical ***entities*** that are rotational symmetrical in a plane, that is, the plane may have a physical reality to our intuition interpretation. The object can be a surface we can see.
We conclude here that a bivector B represents a ***primary quality of second grade*** (***pqg***-2), where the ***quantity*** of the plane segment according to (5.75) and (5.65) has two states

(5.150) $$\mathrm{B} = \pm 1|\mathrm{B}|\boldsymbol{i},$$

Where the real positive scalars $\beta = |\mathrm{B}| \in \mathbb{R}_{+\mathrm{pqg\text{-}2}}$ are the areas of bivector plane segments.
The bivector thus has a bivalent[246] scalar relation to the unit of the bivector per (5.75) and reference to (5.81). – If we multiply by the complex imaginary number unit, we get a pure mathematical imaginary number $i\beta \in \mathbb{C}$. As a subject, it is a bivector $\beta\boldsymbol{i}$ for the plane substance.
Thus, the unit basis $\boldsymbol{i}$ is the ***directive*** generator of a plane-segment rotation.
In the plane, the formula (5.150) can also be written in the linear form, which often is called a pseudoscalar for the plane, but actually is a bivector in space

(5.151) $$\mathrm{B} = \beta\boldsymbol{i} = \boldsymbol{i}\beta, \quad \text{as a plane segment } \textbf{\textit{quantity}} \ \left[\mathbb{R}^1_{\boldsymbol{i},\mathrm{pqg\text{-}2}}\right], \quad \text{where } \beta \in \mathbb{R}_{\mathrm{pqg\text{-}2}} \text{ is a scalar.}$$

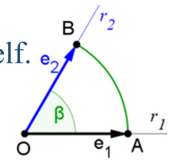
The bivector represents in itself, not a specific geometric figure (object), but just a rotation invariant plane segment with ***direction*** (a subject in the plane substance).
We compare with the complex formulation in formulas (5.137) and (5.138) and have the bivector $\mathrm{B} = \boldsymbol{i}\rho\sin\theta$, which represents the circular invariant plane segment, as seen from the outside of the plane is a ***pqg***-2 ***quantity***, while the term $\rho\cos\theta$ in (5.138) is a pure inner scale, that is a ***pqg***-0 ***quantity*** internal for the plane, which does not represent any spatial ***direction***, and therefore, of course, is circular rotational invariant.
We consider a plane ***entity*** in the space $\mathfrak{G}$ of physics represented by a geometric complex ***quantity***

(5.152) $$\mathcal{Z} = \rho e^{\boldsymbol{i}\theta} = \rho\cos\theta + \boldsymbol{i}\rho\sin\theta, \quad \text{corresponding to the 2-multi-vector } \mathcal{Z} = \mathbf{ba} = \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\wedge\mathbf{a},$$

where all the sequential choices of 1-vectors $\mathbf{a}$ and $\mathbf{b}$ have the relative mutual relationship $|\mathbf{a}||\mathbf{b}| = \rho$ and $\measuredangle(\mathbf{a},\mathbf{b}) = \theta$ in the same plane, and else have no restrictions in that plane. An example of this is illustrated in Figure 5.29, (5.106).
This gives a circular invariant symmetry, which if it represents a real subject of a physical ***entity*** shows a plane $\gamma_{\mathbf{a},\mathbf{b}}$, as an object surface for our intuition of the physical symmetry of that ***entity***.

5.3.4.2. The 2-dimensional Plane

We repeat from Euclid's elements: Quote [12] "
E I.De.7. A *plane surface* is a surface which lies evenly with the straight lines on itself.
E I.De.8. A *plane angle* is the inclination to one another of two lines in
a plane which meet one another and do not lie in a straight line. "



We have previously claimed that a plane is defined from three points, here the intersection O and two other points A and B, which we choose in the distance 1 from O
(a unit radius ρ=1). We have seen that the ***quantity*** of the angle is connected to the arc of the circle, and since the 2-multi-vector $\mathbf{e}_2\mathbf{e}_1 = (\overrightarrow{\mathrm{OB}})(\overrightarrow{\mathrm{OA}})$ is circular invariant in the plane, we conclude that the unit circle is the founding subject of the plane angle substance.
The new idea is that the circle (E I.De.15.) is the a priori figure that generates our intuition of the plane, defined by a uniform radius, (E I.Po.3.) from a center O, like the local origo,

[246] Bivalent refers to the point of view of the intuition of the area relative to the rotation. The sign ± 1 depends not on if the plane area is seen from the front or the back, but on the sequential order for 1-vectors for the sign of $\sin\theta$, which also follows from the orientation of rotation. Often such scalars are called pseudo-scalars for rotation planes. But here we prefer the bivector concept.

(E I.De.16.). The radius concept is associated with the notion of a half-line ray $r_{\mathrm{OP}} \subset \ell_{\mathrm{OP}}$, radiating outward from center O through any point P on the circle and continuing beyond P along the straight line as previously shown in § 4.4.2.7, 4.4.2.8 and later in Figure 5.37.
We introduced an arbitrary radius vector $\mathbf{u} = \overrightarrow{\mathrm{OP}}$, which points to P, and now we write $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, where $u_1, u_2 \in \mathbb{R}$ are coordinates for $\mathbf{u}$ and P in the usual way.
The unit 1-vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{u}$ are radii in the unit circle. To hold and maintain the circle in space $\mathfrak{G}$, we must define at least two points A, and B on the circle in addition to O. This defines two fixed 1-vectors $\mathbf{e}_1 = \overrightarrow{\mathrm{OA}}$ and $\mathbf{e}_2 = \overrightarrow{\mathrm{OB}}$. Based on this circle definition, we can set the judgment $|\mathbf{u}| = |\mathbf{e}_1| = |\mathbf{e}_2| = 1$. The basis set $\{\mathbf{e}_1, \mathbf{e}_2\}$, where $|\mathbf{e}_1| = |\mathbf{e}_2| = 1$ is normalized and is called a normal basis for the coordinate system. The requirement[247] is that $\mathbf{e}_1 \neq \mathbf{e}_2$, as required A≠B, O ∉AB. The two 1-vectors $\mathbf{e}_1, \mathbf{e}_2$ together designate then from the O the plane subject $\gamma_{\mathrm{OAB}} = \gamma_{\bigcirc\mathrm{ABP}} = \gamma_{\bigcirc\mathrm{ABC}}$, through the a priori designation of the two points A,B, which suggests a two-dimensional plane. If we choose one $\mathbf{u}$ of the 1-vectors $\mathbf{u}_{\mathrm{C}} = \overrightarrow{\mathrm{OC}}$, so that $\mathbf{e}_1 \neq \mathbf{u}_{\mathrm{C}} \neq \mathbf{e}_2$, then the three 1-vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{u}_{\mathrm{C}}$ form the circumscribed circle ○ABC.
We judge a priori that several 1-vector always can be 'shifted' as translation invariants, so that, their objects are pointing from a given starting point O as origo. The linear forms

(5.153) $$\mathbf{r} = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 = \overrightarrow{\mathrm{OZ}} = \rho\overrightarrow{\mathrm{OP}} = \rho\mathbf{u} = \rho u_1\mathbf{e}_1 + \rho u_2\mathbf{e}_2, \text{(where } \rho = |\mathbf{r}| \text{ as in section 5.3.2)}$$

only makes sense when we include the *a priori judgment,* that linearly combined 1-vectors $\mathbf{r}$ lie in the plane spanned by the two 1-vectors $\mathbf{e}_1, \mathbf{e}_2$ or the points O,A,B.
Conversely, the three points A, B, C from Figure 5.31 span the object plane $\gamma_{\mathrm{ABC}} = \gamma_{\mathrm{OAB}}$ uniquely by postulate (E XI.De.8), just like the points, span the circumscribed circle ○ABC to triangle △ABC, where the third point C represents a point P on the circle as above. The origo point O in the center of △ABC and thus of ○ABC is the center or the origo from which the circular rotation plane $\gamma_{\bigcirc\mathrm{ABC}}$ is spanned. If the angle ∠AOB can be chosen right angular ⊥ we choose $\mathbf{e}_2\cdot\mathbf{e}_1 = 0$. We will prefer to rename these $\boldsymbol{\sigma}_1 = \mathbf{e}_1$, $\boldsymbol{\sigma}_2 = \mathbf{e}_2$, under the condition $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 = 0$

5.3.4.3. The Orthonormal Reference for the 2-dimensional Plane

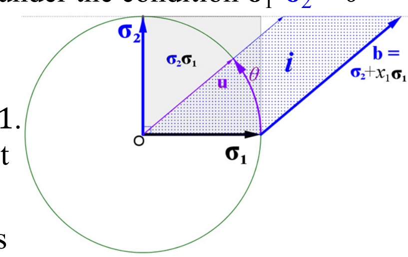


Figure 5.40 $\boldsymbol{i}$
The unit area.

The orthonormal basis set $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$:
Orthogonal $\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_1 = 0$, and normalized $\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = 1 \Rightarrow |\boldsymbol{\sigma}_1| = |\boldsymbol{\sigma}_2| = 1$.
From this concept, we have defined the plane complex ***quantity*** unit as a 2-multi-vector $\boldsymbol{i} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1$.
In that $\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_1 = 0$, therefore it is a bivector $\boldsymbol{i} = \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1 = \mathbf{b}\wedge\boldsymbol{\sigma}_1$ as shown in Figure 5.40, where $\mathbf{b} = \boldsymbol{\sigma}_2 + x_1\boldsymbol{\sigma}_1$, as $\boldsymbol{i} = \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1 = (\boldsymbol{\sigma}_2 + x_1\boldsymbol{\sigma}_1)\wedge\boldsymbol{\sigma}_1$, hence $(\mathbf{b}\wedge\boldsymbol{\sigma}_1)^2 = -1$, and therefore $|\mathbf{b}\wedge\boldsymbol{\sigma}_1| = 1$, and correspondingly
$\boldsymbol{i} = \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\wedge\left(\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2\right) = \boldsymbol{\sigma}_2\wedge\mathbf{c}$, which figuration is left to the reader.
Note, when $x_1 \neq 0$ for 1-vector $\mathbf{b} = \boldsymbol{\sigma}_2 + x_1\boldsymbol{\sigma}_1$ the scalar product $\mathbf{b}\cdot\boldsymbol{\sigma}_1 \neq 0$,[248] hence $\mathbf{b}\boldsymbol{\sigma}_1 = \mathbf{b}\cdot\boldsymbol{\sigma}_1 + \boldsymbol{i}$, similarly, $\mathbf{c} = \boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2$ with $x_2 \neq 0$ gives $\boldsymbol{\sigma}_2\cdot\mathbf{c} \neq 0$, hence $\boldsymbol{\sigma}_2\mathbf{c} = \boldsymbol{\sigma}_2\cdot\mathbf{c} + \boldsymbol{i}$.
The bivector $\boldsymbol{i}$ subject is the same in all the editions $(\boldsymbol{\sigma}_2 + x_1\boldsymbol{\sigma}_1)\wedge\boldsymbol{\sigma}_1$ in Figure 5.40.[249]
This is the *principle* that any Euclidian geometric coordinate system can be made orthogonal.
Put in another way, if we have a unit-1-vector $\boldsymbol{\sigma}_1$ in a given plane ***direction*** $\boldsymbol{i} = \mathbf{b}\wedge\boldsymbol{\sigma}_1$, we can always find another 1-vector $\boldsymbol{\sigma}_2 = \mathbf{b} - x_1\boldsymbol{\sigma}_1$ fulfilling $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 = 0$ so that $\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \boldsymbol{i}$ is the plane unit.
The orthonormal basis set $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$ is often advantageous as a cartesian intuition.

[247] Negation: A geometrical subject that contains only parallel lines E I.De.23 is not a two-dimensional plane.
[248] Anyway, the scalar $\mathbf{b}\cdot\boldsymbol{\sigma}_1 = \cos\theta$ express the non-orthogonality measure of the covariant parallel structure.
[249] It is mentioned as a curiosity that in Figure 5.40 is $|\mathbf{b}| = |1/\sin\theta|$ and $x_1 = \cot\theta$, as well as the other $x_2 = \tan\theta$.
In addition, note that from the equation $\mathbf{b}\wedge\boldsymbol{\sigma}_1 = \boldsymbol{i}$, we cannot conclude orthogonality between the 1-vectors.