and subsequent addition through line combinations can form all possible 1-vectors in space $\mathfrak{G}$. Hereby we judge that the *Cartesian* vector algebra is of ***primary quality of first grade*** (***pqg***-1). The *Cartesian coordinate* system historically had significant importance for the concept of space $\mathfrak{G}$. (especially up to three dimensions) for an understanding the location of physical objects in space $\mathfrak{G}$, but it has been problematic for the understanding of the form of physical objects, as the idea of the perpendicular (orthogonal) axes has camouflaged the plane rotation as a property of ***entities*** in space. The 4-symmetry between the two perpendicular axes with four quadrants is shown in Figure 5.36 in four even angles leaves one to believe that an alternation of the two axes is indifferent to Leibniz rational relation of spatial forms. This was first contradicted by Kant 1768 [11] p.361-372.

One could believe that line segment AB is the same as line segment BA in Figure 5.35.

When we see a third point $0 \notin \ell_{AB}$ outside the line, the line suddenly gets a front- and back-side, namely the front and back of the plane idea for us formed by the three points γ_{OAB}.

In short above (5.73), we have formulated this with the anti-commuting law

(5.140) $$\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$$

This means that a permutation of the coordinate axes will change the sign of a plane segment. Hence a plane segment has two states, as indicated by (5.75) the bivector $\hat{\mathrm{B}} = \pm 1\boldsymbol{i}$ or $\mathrm{B} = \pm\beta\boldsymbol{i}$.

Saying in an everyday way the reason is: A piece of paper has both a front and a back, and a clock turns counterclockwise when viewed from the backside (imagine the dial is transparent).

A rotation ***direction*** has two orientations OAB or OBA. On top of this we have the problem: What is clockwise or counterclockwise rotation depending on the point of view.

5.3.3.2. The Cartesian Coordinate System and the Plane Pseudoscalar Concept

We look at points in the *Cartesian* coordinate system described in formula (5.135)-(5.139)

(5.141) $$Z \leftrightarrow \mathrm{X} \leftrightarrow \mathbf{r} = \mathbf{r} = \overrightarrow{0\mathrm{X}} = \overrightarrow{0\mathrm{X}}_1 + \overrightarrow{0\mathrm{X}}_2 = x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{i}\boldsymbol{\sigma}_1 = (\rho\cos\theta)\boldsymbol{\sigma}_1 + (\rho\sin\theta)\boldsymbol{\sigma}_2 = \rho e^{\boldsymbol{i}\theta}\boldsymbol{\sigma}_1.$$

Multiplying from the right with $\boldsymbol{\sigma}_1$, we get the 2-multi-vector concept

(5.142) $$Z \leftrightarrow \mathcal{Z} = \rho U_\theta = \rho e^{\boldsymbol{i}\theta} = \mathbf{r}\boldsymbol{\sigma}_1 = \mathbf{r}\cdot\boldsymbol{\sigma}_1 + \mathbf{r}\wedge\boldsymbol{\sigma}_1 = \rho\cos\theta + \rho\,\boldsymbol{i}\sin\theta = x_1 + x_2\boldsymbol{i}$$

In the last three formulations in (5.142), the first part in sum is a real scalar x_1, and the second part is a unit bivector $\boldsymbol{i}$ times a real scalar x_2 or what is called a ***pqg***-2 pseudo-scalar for the plane, pseudonym with a plane bivector $\mathrm{B} = x_2\boldsymbol{i}$. The reader can compare it with § 5.1.1.10.

Instead of considering the plane as given *Cartesian* coordinates (x_1, x_2) we by (5.142) try to intuit the plane as given by two ***primary quantities***

- ***pqg***-0 a real scalar (x_1),
- ***pqg***-2 a bivector $(x_2\boldsymbol{i})$ (as an orientation-dependent pseudoscalar in the same plane).

These can be added in geometric algebra, although they have two different ***qualities*** in the substance of space $\mathfrak{G}$. This concept depends fully on an object $\boldsymbol{i} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ for intuition.

Anyway, we have a linear algebraic space, which I try to describe

$(\mathbb{R}_1^1 \otimes \mathbb{R}_1^1, \mathbb{R}) \sim (\mathbb{R}^2, \mathbb{R}) \sim (\mathbb{R}_{\text{pqg-0}} \otimes \mathbb{R}_{\text{pqg-2}}^{\boldsymbol{i}}, \mathbb{R}) \sim (\mathbb{R}_1^1 \oplus \mathbb{R}_{\boldsymbol{i}}^1, \mathbb{R}) \sim (\mathbb{R}_1^1 \oplus \mathbb{R}_1^1, \mathbb{R}) \sim (\mathbb{R}_1^2, \mathbb{R})$

It is left here for the reader to consider the enlightenment connection with the coordinates

$Z \leftrightarrow \mathbf{r}\boldsymbol{\sigma}_1 \leftrightarrow (\rho, \theta) \leftrightarrow (\rho, \boldsymbol{i}\theta) \rightleftarrows (x_1, x_2\boldsymbol{i}) \leftrightarrow (x_1, x_2) \leftrightarrow \mathrm{X}.$

5.3.3.3. The Parity Inversion of the 2-dimensional Descartes Extension Coordinate

A *parity inversion by first grade* ***pqg***-1-vectors, with impact on higher ***grade*** elements, is that Hestenes call space conjugation in [6]p.17. We define it by orientation inverting of all 1-vector ***directions*** units of Descartes extension inside the physical space $\mathfrak{G}$. In the Cartesian plane, any 1-vector is generated from the two base-1-vectors according to the formula (5.136)

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(5.143) $$\mathbf{r} = x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2.$$

As we have described in formula (5.121) and earlier in § 4.4.2.5 (4.61) the ***pqg***-1 parity of each coordinate axis may be described as $U_\pi\boldsymbol{\sigma}_1 = \boldsymbol{ii}\boldsymbol{\sigma}_1 = -1\boldsymbol{\sigma}_1$ and $U_\pi\boldsymbol{\sigma}_2 = \boldsymbol{ii}\boldsymbol{\sigma}_2 = -1\boldsymbol{\sigma}_2$ for $\forall\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$.

Similarly, (5.123) any linear 1-vector has a parity inversion transformation of the form

(5.144) $$\overline{\mathbf{r}} = U_\pi\mathbf{r} = \boldsymbol{ii}(x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2) = -(x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2) = -\mathbf{r},$$ (see Figure 5.34)

The parity-operator $\overline{\mathbf{a}} = \boldsymbol{ii}\mathbf{a} = U_\pi\mathbf{a} = -1\mathbf{a}$ is a ***pqg***-1 operation, which always multiplies **all** possible line segment ***pqg***-1-vectors in space $\mathfrak{G}$ with the scaling -1.

5.3.3.4. The Extension *Grade One* Parity Inversion of Scalars and Bivectors

All scalars are invariant to the parity-inversion-operation by the a priory synthetic judgment

(5.145) $$\overline{\lambda} = \lambda \in \mathbb{K}$$

For the scalar product of 1-vectors, we too have invariance by ***grade one*** parity inversion

(5.146) $$\overline{\mathbf{b}\cdot\mathbf{a}} = \overline{\mathbf{b}}\cdot\overline{\mathbf{a}} = (\boldsymbol{ii}\,\mathbf{b})\cdot(\boldsymbol{ii}\,\mathbf{a}) = (\boldsymbol{iiii})\mathbf{b}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b},$$

All kind of ***pqg***-0 scalars is invariant to parity operations!

When it comes to a plane ***pqg***-2 parity-inversion-operation on a bivector we have

$\overline{\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1} = (U_\pi\boldsymbol{\sigma}_2)(U_\pi\boldsymbol{\sigma}_1) = (\boldsymbol{ii}\boldsymbol{\sigma}_2)(\boldsymbol{ii}\boldsymbol{\sigma}_1) = (-\boldsymbol{\sigma}_2)(-\boldsymbol{\sigma}_1) = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1,$ as illustrated in Figure 5.36.

When the parity-operator works on any bivector $\mathbf{B}$, that can always be dissolved as $\mathbf{B} = \mathbf{b}_2\mathbf{b}_1$ with $\mathbf{b}_2\cdot\mathbf{b}_1 = 0$, which has the consequence

(5.147) $$\overline{\mathbf{B}} = \overline{\mathbf{b}_2\mathbf{b}_1} = (U_\pi\mathbf{b}_2)(U_\pi\mathbf{b}_1) = (\boldsymbol{ii}\mathbf{b}_2)(\boldsymbol{ii}\mathbf{b}_1) = (-\mathbf{b}_2)(-\mathbf{b}_1) = \mathbf{b}_2\mathbf{b}_1 = \mathbf{B},$$

and in general, invariant for any bivector $\mathbf{B} = \mathbf{a}\wedge\mathbf{b}$,

(5.148) $$\overline{\mathbf{B}} = \overline{\mathbf{b}\wedge\mathbf{a}} = (U_\pi\mathbf{b})\wedge(U_\pi\mathbf{a}) = (\boldsymbol{ii}\mathbf{b})\wedge(\boldsymbol{ii}\mathbf{a}) = -\mathbf{b}\wedge-\mathbf{a} = \mathbf{b}\wedge\mathbf{a} = \mathbf{B},$$

and for a 2-multi-vector product of two 1-vectors $A = \mathbf{ba} = \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\wedge\mathbf{a}$.

(5.149) $$\overline{\mathbf{ba}} = \overline{\mathbf{b}}\overline{\mathbf{a}} = (U_\pi\mathbf{b})(U_\pi\mathbf{a}) = (\boldsymbol{ii}\mathbf{b})\cdot(\boldsymbol{ii}\mathbf{a}) + (\boldsymbol{ii}\mathbf{b})\wedge(\boldsymbol{ii}\mathbf{a}) = -\mathbf{b}(-\boldsymbol{a}) = \mathbf{ba},$$

Both these ***pqg***-0 and ***pqg***-2 subjects are invariant to parity-inversion-operation! It is fully consistent with the fact that a rotation in the bivector plane does not change the bivector.

Here I mention as we will see that all higher ***primary quality even grades*** are invariant to parity-inversion-operation while ***odd grades*** have reversed orientations after a parity-inversion-operation. In all, the idea of parity-inversion-operation as a creation action is creasy, because it never can have an impact in real physics (*you* cannot inverse it at all), but it is good as an idea for balancing in a concept of Newton's third law, and further help us understand and interpret the geometric structure of $\mathfrak{G}$ space in physics. The definition of parity-inversion-operation $\overline{\mathbf{a}} = \boldsymbol{ii}\mathbf{a}$ is the origo of ***pqg***-1-vectors by an extreme special U_π case of the 1-rotor concept U_θ where the rotation plane collapses to the ***direction*** of the 1-vector on which it acts and thus alternates its orientation.

In contrast, we have $U_\pi(\mathbf{B}) = \mathbf{B}$ as any rotation of form $U_\theta(\mathbf{B}) = \mathbf{B}$ in the bivector plane will maintain the bivector, as generally described in §5.2.8, $U(\mathbf{b}\wedge\mathbf{a}) = \mathbf{b}\wedge\mathbf{a}$, shown in Figure 5.27 where the reasoning was geometric. This of course also applies to the plane ***pqg***-2 substance in the extreme case (parity-inversion operation) $\boldsymbol{ii}(\mathbf{B}) = \boldsymbol{ii}(\mathbf{b}\wedge\mathbf{a}) = \mathbf{b}\wedge\mathbf{a} = \mathbf{B}$. – [245]

The even plane concept is parity invariant! And of course, auto-rotation invariant. $U_\pi(U_\theta) = U_\theta$.

Anyway, the reversion of even bivectors $\mathbf{B} = \mathbf{b}\wedge\mathbf{a}$ is the reversed operation order of the 1-vectors $\mathbf{B}^\dagger = -\mathbf{B} = \mathbf{a}\wedge\mathbf{b}$.

[245] Once again it is important here to note the notation with the $\boldsymbol{ii}(0)$, as the parity-inversion-operation works on all line 1-vector ingredients in the 2-multi-vector, thus $\boldsymbol{ii}(\boldsymbol{i}) = \boldsymbol{ii}(\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1) = \boldsymbol{ii}(\boldsymbol{\sigma}_2)\boldsymbol{ii}(\boldsymbol{\sigma}_1) = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \boldsymbol{i}$ has no effect, while when we simply multiply the unit bivectors we have $\boldsymbol{iii} = \boldsymbol{i}^3 = -\boldsymbol{i}$, and when the bivector $\boldsymbol{i}^3$ acts on a 1-vector like in (5.77) we get $\boldsymbol{i}^3\boldsymbol{\sigma}_1 = (\boldsymbol{ii})\boldsymbol{i}\boldsymbol{\sigma}_1 = \boldsymbol{ii}(\boldsymbol{\sigma}_2) = -\boldsymbol{\sigma}_2$.