5.2.8.2. The Product of Rotors

The product of 1-rotors is equivalent to the addition of the angles in the same plane

(5.94) $\quad U_\theta U_\varphi = U_{\theta+\varphi} \quad$ or $\quad e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$

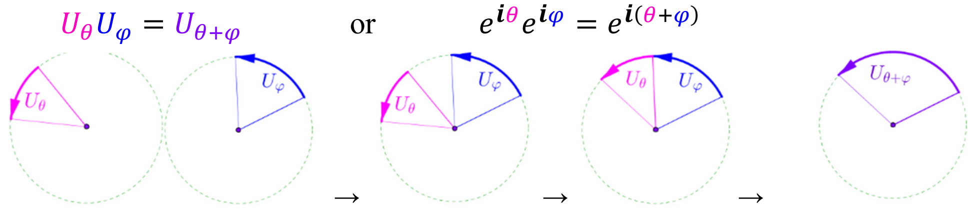

Figure 5.25 Plane product of two 1-rotors ↕ Addition of the arc ***quantities*** of the angles in the same plane.

5.2.8.3. The Rotor Product With a 1-vector

Multiplying a 1-vector with a rotor from the left or right side switches the 1-vector ***direction*** in the rotor plane, as illustrated in Figure 5.26,

(5.95) $\quad U_\theta \mathbf{a} = e^{i\theta}\mathbf{a} = \mathbf{b} \quad = \quad \mathbf{a}e^{-i\theta} = \mathbf{a}U_\theta^\dagger = \mathbf{b} \qquad$ or $\quad \mathbf{a}U_\theta = \mathbf{a}e^{i\theta} = \mathbf{c} = U_\theta^\dagger\mathbf{a} = e^{-i\theta}\mathbf{a}$

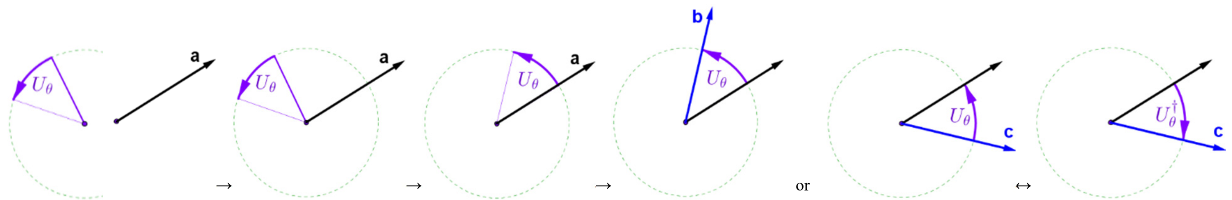

Figure 5.26 The rotor operator acts on a 1-vector from left or from right: $U_\theta\mathbf{a} = \mathbf{b} \quad \leftrightarrow \quad \mathbf{a}U_\theta = \mathbf{c} = U_\theta^\dagger\mathbf{a} = \mathbf{c}$.

5.2.8.4. A Bivector is Self-identic by a Rotation in its own Plan

It is important to note that the rotation of a rotor in its plane is auto-identical $U_\theta\boxed{U_\varphi} = U_\varphi$.

I have used the box $\square$ as a symbol of the operation in the same plane instead of brackets as we use for multiplication in that same plane (5.94) $U_\theta(U_\varphi) = U_{\theta+\varphi}$. Further is noted that the rotation of a bivector $\mathbf{b}\wedge\mathbf{a}$ through its plane $\gamma_{\mathbf{a,b}}$ with any arbitrary rotor do not impact a bivector or either a scalar product $U\boxed{\mathbf{b}\wedge\mathbf{a}} = (U\mathbf{b})\wedge(U\mathbf{a}) = \mathbf{b}_\prime\wedge\mathbf{a}_\prime = \mathbf{b}\wedge\mathbf{a}$ or $U\boxed{\mathbf{b}\cdot\mathbf{a}} = (U\mathbf{b})\cdot(U\mathbf{a}) = \mathbf{b}_\prime\cdot\mathbf{a}_\prime = \mathbf{b}\cdot\mathbf{a}$

The same applies to 2 multi-vectors $U\boxed{\mathbf{ba}} = (U\mathbf{b})(U\mathbf{a}) = \mathbf{b}_\prime\mathbf{a}_\prime = \mathbf{ba}$, in that $\mathbf{ba} = \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\wedge\mathbf{a}$.[232] The rationale here is the geometric congruence between 1-vectors and their mutual angles; algebraic reasoning will follow later below. It is noted that the same of course applies to rotor translations as in § 5.2.7.4.

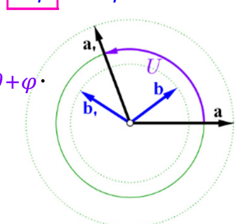

Figure 5.27 Rotation invariance.

This invariance of the rotor implies that the intuition of a center object for a rotation is irrelevant to the rotating substance of the concept of a plane ***direction***. This is characteristic of the ***categorical*** idea of a ***primary quality of second grade*** (***pqg***-2).

5.2.8.5. The Simple 1-rotor Algebra

The formulas (5.94) and (5.95) represent the multiplication algebra for the simple rotors. The rotor is just the operator, which defines a plane and therefore a geometric multiplication algebra for planes. In one plane given by the ***direction*** idea $\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$, we have the rotor idea (5.83)

(5.96) $\quad U_\theta \coloneqq e^{+i\theta}$

It is purely given by *one* angular parameter argument θ in the *one* and lonely plane ***direction i***, therefor we call this simple rotor for a 1-rotor. We recall from (5.85) and (5.87) that a rotor always has a unit magnitude modulus amplitude $|U_\theta| = \sqrt{\langle U_\theta^\dagger U_\theta\rangle_0} = 1$.

[232] This operator impact can also be noted by the dot notation $\dot{U}\mathbf{b}\dot{\mathbf{a}} = (U\mathbf{b})(U\mathbf{a})$ on both operators; with other operands than the first.

5.2.9. A Complex *Quantity* in Space Called a Plane Spinor

5.2.9.1. The Complex *Quantity* as a Geometric Product of Two 1-vectors

The idea of the 1-rotor as a complex exponential function $U_\theta = e^{i\theta}$ representing a rotation ***direction*** along with the unit circle $\overrightarrow{\text{arc}}$, may by *dilation* with a real scalar ρ and an appropriate choice of θ be constructed to represent any complex multivector ***quantity***

(5.97) $\quad \mathcal{Z} = \rho e^{+i\theta} = \rho U_\theta = \mathbf{ba}$

It is funny that $\mathbf{a}$ and $\mathbf{b}$ can be arbitrary[233] as long as $|\mathbf{ba}| = \rho$ and $\theta = \sphericalangle(\mathbf{a},\mathbf{b})$. The complex conjugate is equivalent to a reverse rotation

(5.98) $\quad \mathcal{Z}^\dagger = \rho e^{-i\theta} = \rho U_\theta^\dagger = \mathbf{ab}$

These complex ***quantities*** are equivalent to the ($\overrightarrow{\text{arc}}$) of ***direction*** magnified by *dilation*. The product of these two gives

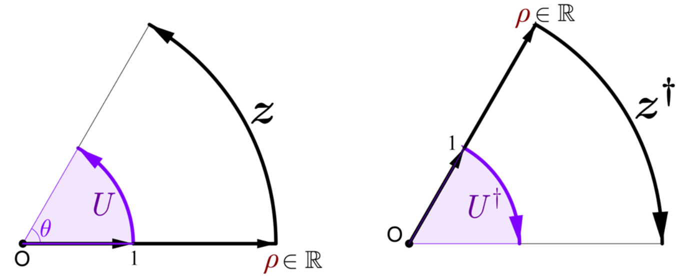

Figure 5.28 Two mutually reverse complex ***quantities***. ($\mathbf{a}$ and $\mathbf{b}$ not shown).

(5.99) $\quad \mathcal{Z}\mathcal{Z}^\dagger = (\mathbf{ba})(\mathbf{ab}) = \mathbf{a}^2\mathbf{b}^2 = |\mathcal{Z}|^2 = \rho U_\theta \rho U_\theta^\dagger = \rho^2 \in \mathbb{R}_{+\text{pqg-0}}$

The real scalar $\rho = |\mathcal{Z}| = |\mathbf{a}||\mathbf{b}| \in \mathbb{R}_+$ is called the *modulus amplitude* of the complex ***quantity*** $\mathcal{Z}$. This amplitude[234] is completely independent of the 1-rotor $U = e^{i\theta}$ and thus the angle θ of rotation in the unit circle. The unitary rotor U havs modulus $|U| = 1$, since $U_\theta U_\theta^\dagger = e^{i\theta}e^{-i\theta} = e^{i0} = 1$, see (5.85).

Compare to the unit bivector $\boldsymbol{i}$ of the plane ***direction*** whose amplitude is $|\boldsymbol{i}| = 1$, as

(5.100) $\quad \boldsymbol{i}\boldsymbol{i}^\dagger = \boldsymbol{i}(-\boldsymbol{i}) = 1$.

With the intuitive explanation $\boldsymbol{i}\boldsymbol{i}^\dagger = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = 1$ from a basis set $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$.

Remember here that the operator $\boldsymbol{i}$ must be used four times to get the modulo unit

(5.101) $\quad (\boldsymbol{iiii}) = \boldsymbol{i}^4 = 1$.

This entails that n multiple operations $(\boldsymbol{iiii})^n$ represent the modulo of n cyclic turns in the $\boldsymbol{i}$ plane of the *complex* ***quantity*** a 2-multi-vector, that as an intuit object can be a geometrical product of two 1-vectors, here represented as $\mathbf{b}$ and $\mathbf{a}$.

(5.102) $\quad \mathcal{Z} = \rho e^{+i(\theta+2\pi n)} = \rho(\boldsymbol{iiii})^n e^{i\theta} = \rho\, \boldsymbol{i}^{4n} e^{i\theta} = (\boldsymbol{iiii})^n\mathbf{ba} = \mathbf{ba}, \quad$ for $n \in \mathbb{Z}$

This ($\overrightarrow{\text{arc}}$) is multiple additive arguments with modulo a full circle ○ circumference with the real scalar ***quantity*** 2π, whereby all the $\theta + 2\pi n$ is the angular argument for the same rotor

(5.103) $\quad \hat{\mathcal{Z}} = U_\theta = e^{+i\theta} = \widehat{\mathbf{ba}}, \quad$ where $\quad |\hat{\mathcal{Z}}| = |\widehat{\mathbf{ba}}| = |\hat{\mathbf{b}}||\hat{\mathbf{a}}| = 1, \quad$ especially $\mathbf{u}_2 = \hat{\mathbf{b}}$ and $\mathbf{u}_1 = \hat{\mathbf{a}}$

The complex ***quantity*** $\mathcal{Z}$ is composed of two ***quantities***

- the ***quantity*** $[\mathbb{R}_{+\text{pqg-0}}]$ of the real scalar *modulus amplitude*, $|\mathcal{Z}| = \rho \in \mathbb{R}_+$, making the dilation, and
- the ***quantity*** $[\mathbb{R}^1_{i,\text{pqg-2}}]$ one unitary *rotor* $U_\theta = e^{i\theta} = \mathbf{u}_2\mathbf{u}_1$ (5.83) as a geometric unitary product between two 1-vectors with one mutual angle $\theta = \sphericalangle(\mathbf{u}_1, \mathbf{u}_2)$, parameterised by one periodic real scalar $\theta \in \mathbb{R}$ modulo 2π.

This combined product ***quantity*** ρU_θ is called a 1-*spinor* for the plane (one angular parameter). The 1-spinor is a ***primary quality of zero and second grade*** (***pqg***-0-2), i.e., even ***grades***.

[233] The 1-vector concept is hidden as subject to intuition in this complex construction of a ***pqg***-0,2 rotor.

[234] Some omit amplitude for modulus, but in spoken language, it can then be confused with modulo see below for the return (5.101) by the argument $\theta + 2\pi n$ in (5.102). In this way, I prefer amplitude, as it is often used for signals, etc.