5.2.7.2. Intuition of the Bivector and the Scalar for the Interpretation of a Rotation

These rotations have a ***direction*** along the arc of the unit circle, a circular ***direction*** ($\overrightarrow{\text{arc}}_\theta$).
The rotor U_θ is equivalent to Euler's formula of the unit circle in which the circle arc angle $\theta \in \overrightarrow{\mathbb{R}}$ ***quantity*** is a measure. It is an objective argument in the function $\sin\theta$ defined as the measure of the arc height, which is the height of the parallelogram formed by the ***grade***-2 bivector $\mathbf{u}_2\wedge\mathbf{u}_1$, and thus the area of the parallelogram with a unit baseline, see Figure 5.20.
While the function $\cos\theta$ is defined as the ***grade***-0 measure from the center to the projection on the opposite 1-vector as the scalar product $\mathbf{u}_2\cdot\mathbf{u}_1$, i.e., the co-linear magnitude (co-sinus).

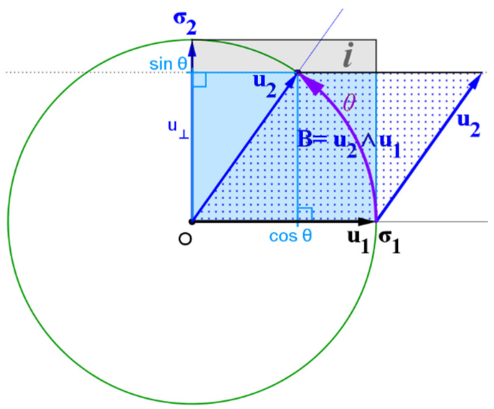
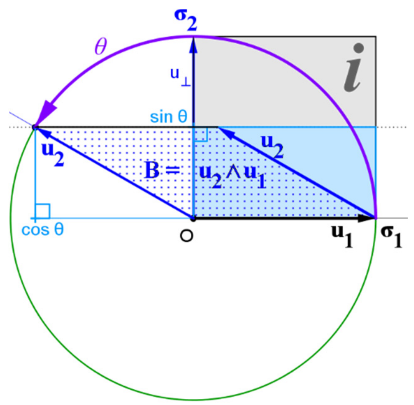


Figure 5.20
← Two cases of θ →
The bivector $\mathbf{B}=\mathbf{u}_2\wedge\mathbf{u}_1$,
where $\theta = ∢(\mathbf{u}_2,\mathbf{u}_2)\in\overrightarrow{\mathbb{R}}$,
$|\mathbf{u}_1|=|\mathbf{u}_2| = 1 \in \mathbb{R}$,
$|\mathbf{u}_\perp|=|\boldsymbol{i}|= 1$ and $\boldsymbol{i}^2 = -1$.



- The bivector $\mathrm{B}=\mathbf{u}_2\wedge\mathbf{u}_1=\boldsymbol{i}\sin\theta$ constitute the area-segment, generated by the angle $\angle(\mathbf{u}_1,\mathbf{u}_2)$ in that plane ***direction*** given by the unit area-segment $\boldsymbol{i} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ multiplied by the relative height of the arc: $\sin\theta$ by the angle rises ⊥ rejection from the primary baseline 1-vector $\mathbf{u}_1$. The scalar $\mathbf{u}_2\cdot\mathbf{u}_1 = \cos\theta \in\mathbb{R}$ represents the relative co-linear magnitude part that is generated by the angle $\angle(\mathbf{u}_1,\mathbf{u}_2)$ along one of the 1-vectors $\mathbf{u}_1$ or $\mathbf{u}_2$. (See also Figure 5.9)

These two components; the area segment plus the scalar part complement each other in the rotor (5.83), where the angle ***quality*** $\overrightarrow{\text{arc}}_\theta \sim \angle(\mathbf{u}_1,\mathbf{u}_2)$ through the real ***quantity*** $\theta = ∢(\mathbf{u}_1,\mathbf{u}_2)\in\overrightarrow{\mathbb{R}}$ is included as a real argument in the rotor

(5.86) $$\overrightarrow{\text{arc}}_\theta \to U_\theta = e^{\boldsymbol{i}\theta} = \cos\theta + \boldsymbol{i}\sin\theta = \mathbf{u}_2\mathbf{u}_1 = \mathbf{u}_2\cdot\mathbf{u}_1 + \mathbf{u}_2\wedge\mathbf{u}_1 = \mathbf{u}_2\cdot\mathbf{u}_1 + \mathrm{B} = \langle U_\theta\rangle_0 + \langle U_\theta\rangle_2$$

The rotor is a Scalar + Bivector, representing a multi-vector of the form $A = \langle A\rangle_0 + \langle A\rangle_2$, where the scalar $\langle A\rangle_0$ is the real $\mathbb{R}_{\text{pqg-0}}$ ***quantity*** and $\langle A\rangle_2$ is a $\mathbb{R}^1_{i,\text{pqg-2}}$ ***quantity***.
The magnitude, also called the *modulus* of this multi-vector is $|A| = \sqrt{\langle A^\dagger A\rangle_0}$ so that

(5.87) $$|A|^2 = \langle A^\dagger A\rangle_0 = |\langle A\rangle_0|^2 + |\langle A\rangle_2|^2,$$ see later below section 5.3.8.

For the rotor (5.83)(5.85) this is $U^\dagger U = \langle U^\dagger U\rangle_0 = |\langle U\rangle_0|^2 + |\langle U\rangle_2|^2 = (\cos\theta)^2 + (\sin\theta)^2 = 1$.

5.2.7.3. The Plane-segment Unit

The finite rotor $U \neq 1$, different from the identical operator, has a fundamental unit for a ***pqg***-2 plane segments: – Looking at the perpendicular rotor
$U_\perp = \langle U_\perp\rangle_0 + \langle U_\perp\rangle_2 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1 = 0 + \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \boldsymbol{i} = \boldsymbol{i}$,
which is the simplest ***pqg***-2 bivector ***direction*** in the rotation plane.

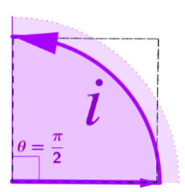


Figure 5.21 The rotor $U_\perp$, where the unit sector area has a radius of $4/\pi$.

We note the magnitude of $\boldsymbol{i}$ is normalized $|\boldsymbol{i}|=1$ and the belonging finite area of the plane segment which we set to 1, although the unit circle quadrant object angle sector area is $\pi/4$, we multiply that with a sector radius $4/\pi$, and we have 1. The reversed-orientated area of $-\boldsymbol{i}$ is -1, refer to (5.75).
The rotor $U_\perp = \boldsymbol{i}$ turns everything in the plane 90° counterclockwise $\sim\theta = \pi/2$, that is,

(5.88) $$\boldsymbol{i} = e^{i\pi/2}$$

Here we remember that the bivector subject is a ghost as a plane amoeba shown in Figure 5.14.
The bivector as object $\boldsymbol{i} = U_\perp = (\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1)$ defines the rotation plane as a specific ***pqg***-2 plane ***direction*** in space. Likewise, the same plane subject is spanned of the two linearly independent 1-vectors $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\}$ by formula (5.23) $\mathbf{y} = \alpha\boldsymbol{\sigma}_1 + \beta\boldsymbol{\sigma}_2$ to all 1-vector objects in that plane. This is where the two distinct ***pqg***-1 ***directions*** are spanning plane $\gamma_{\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\}}$.

5.2.7.4. Rotor Independence of any *pqg*-1 *Direction* in a Plane *Quality*

In general, the rotor U is independent of all linear ***directions*** in space.
As 1-vector objects **a** with the line segment ***direction*** independent of all starting points, thereby translation invariant as shown in Figure 5.22, the rotor U is independent of any start 1-vectors, see Figure 5.23.

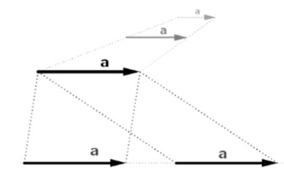


Figure 5.22 One 1-vector **a**.

A rotor U is defining a rotational plane with a ***direction*** and a rotation corresponding to the $\overrightarrow{\text{arc}}$ ***direction*** in its unit circle.
A rotor object with a thought center is translation invariant.
It is here noted that the rotor subject is not only a translation invariant but also rotation plane invariant, as shown in Figure 5.23.
The 1-rotor U_θ as an operator of one angle[229] θ is equivalent to a complex exponential function whose argument is a bivector

(5.89) $$U_\theta := e^{\boldsymbol{i}\theta}$$

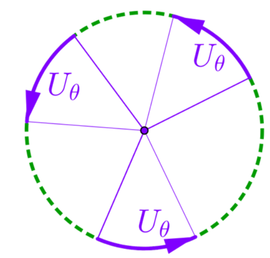


Figure 5.23 One 1-rotor in the plane 𝔓. Three separate objects for that one 1-rotor subject are illustrated for intuition.

We remember $\boldsymbol{i}$ and thus $\boldsymbol{i}\theta = \theta\boldsymbol{i}$ is a bivector,
in that θ is a real scalar for the angle $\overrightarrow{\text{arc}}_\theta$. By this we also assume an order of the scalar $\theta\in\overrightarrow{\mathbb{R}}$ according to the plane $\boldsymbol{i}$ ***direction*** of rotation.

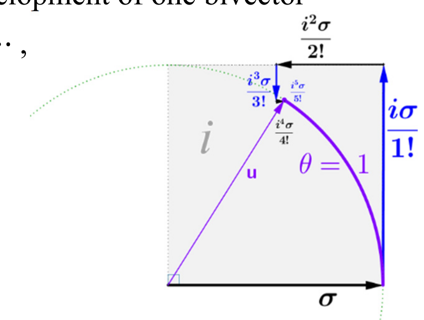
5.2.8. The Exponential Function with one plane *direction* Bivector as Argument

We note that the 1-rotor[230] exponential function is a serial development of one bivector

(5.90) $$e^{\boldsymbol{i}\theta} = \exp(\boldsymbol{i}\theta) = 1 + \frac{(\boldsymbol{i}\theta)}{1!} + \frac{(\boldsymbol{i}\theta)^2}{2!} + \frac{(\boldsymbol{i}\theta)^3}{3!} + \frac{(\boldsymbol{i}\theta)^4}{4!} + \frac{(\boldsymbol{i}\theta)^5}{5!} + \cdots,$$

which therefore is a 2-multi-vector or just a 2-vector.[231]
When we try $\theta = 1$, we get one intuit object

(5.91) $$\mathbf{u}\boldsymbol{\sigma} = e^{\boldsymbol{i}1} = 1 + \frac{\boldsymbol{i}}{1!} + \frac{\boldsymbol{i}^2}{2!} + \frac{\boldsymbol{i}^3}{3!} + \frac{\boldsymbol{i}^4}{4!} + \frac{\boldsymbol{i}^5}{5!} + \cdots,$$



this series of object operators are illustrated in Figure 5.24 as a multiplication operation to a start 1-vector $\boldsymbol{\sigma}$.
Here in the plane substance $\exp(\boldsymbol{i}\theta)$ is *scalar* + *bivector* since $\boldsymbol{i}\theta$ in all the serial parts lay in the same plane subject as $\boldsymbol{i}$,

(5.92) The scalar $$\langle U_\theta\rangle_0 = \langle e^{\boldsymbol{i}\theta}\rangle_0 = \cos\theta = 1 + \frac{(\boldsymbol{i}\theta)^2}{2!} + \frac{(\boldsymbol{i}\theta)^4}{4!} + \cdots$$ and

(5.93) The bivector $$\langle U_\theta\rangle_2 = \langle e^{\boldsymbol{i}\theta}\rangle_2 = \boldsymbol{i}\sin\theta = +\frac{(\boldsymbol{i}\theta)}{1!} + \frac{(\boldsymbol{i}\theta)^3}{3!} + \frac{(\boldsymbol{i}\theta)^5}{5!} + \cdots$$

Figure 5.24 2-vector $\mathbf{u}\boldsymbol{\sigma}$: The exponential series for $\theta = 1$, over the bivector $\boldsymbol{i}$ from a 1-vector $\boldsymbol{\sigma}$ to another 1-vector $\mathbf{u} = e^{\boldsymbol{i}}\boldsymbol{\sigma}$.
$\mathbf{u} = (\mathbf{u}\boldsymbol{\sigma})\boldsymbol{\sigma} = e^{\boldsymbol{i}\theta}\boldsymbol{\sigma}$, (5.90).
We see that the two 1-vector ***directions*** $\boldsymbol{\sigma}$ and $\boldsymbol{i}\boldsymbol{\sigma}$ give terms with alternating orientations.

In Figure 5.24 the 1-vector object $\boldsymbol{\sigma}$ is scaled with the scalar$_0$, while the bivector$_2$ too scales with the rotation perpendicular ⊥ to $\boldsymbol{\sigma}$.
Note that the series (5.92) and (5.93) have alternating orientations in their terms, in that $(\boldsymbol{i}\theta)^2 = -\theta^2$.

[229] The term U_θ with argument θ as indices the rotor is in the plane for the angle, i.e., 2-dimensional (2D), a 1-rotor ***pqg***-2 ***direction***.
[230] The designation 1-rotor refers to an operation in one and the same plane defined by only one $\boldsymbol{i}$ ***direction***, for every $\theta\boldsymbol{i}$ bivector. In chapter 6 we will tread the interconnection of several independent plane ***directions*** in natural space, (by that also 2-rotors).
[231] The blue odd exponents are bivectors, and the black even are scalars. Both multiplied a 1-vector, which gives a 1 vector.

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