This leads to, that the area segment of a subject bivector $\mathbf{B}$ in principle is a *pseudoscalar*[220] for the plane, even though the object $\mathbf{b}\perp\mathbf{a}$ rectangle has a scalar area magnitude

(5.67) $|\mathbf{B}| = |\mathbf{b}||\mathbf{a}| \in \mathbb{R}_+ \geq 0.$

We say that the unit bivector $\widehat{\mathbf{B}}$ is the *unit pseudoscalar* for that plane it defines.

5.2.5.4. A Bivector Multiplied by a 1-vector

First, a bivector is defined as its resolution of two 1-vectors $\mathbf{B}=\mathbf{b}\wedge\mathbf{a}$, that in the tradition span the plane concept substance. In that plane, an outer product of three 1-vectors vanish $\mathbf{b}\wedge\mathbf{a}\wedge\mathbf{c} = 0$. Therefor $\mathbf{B}\wedge\mathbf{c} = 0$ express that any relevant 1-vector $\mathbf{c}$ is internal in the plane spanned by $\mathbf{B}$. [221]

A bivector $\mathbf{B}$ anticommute in multiplying by any each 1-vector in its plane

(5.68) $\mathbf{Ba} = -\mathbf{aB},$

the reason is, there exist a $\mathbf{b}$ using (5.66) $\exists\mathbf{b}\cdot\mathbf{a} = 0 \Rightarrow \mathbf{B} = \mathbf{ba} \Rightarrow \mathbf{Ba} = \mathbf{baa} = -\mathbf{aba} = -\mathbf{aB}$.
This product in a plane is a 1-vector $\mathbf{Ba} = \mathbf{ba}^2 \implies \mathbf{b} = \mathbf{Ba}/\mathbf{a}^2$
The multiplicative inverse 1-vector from (4.76)

(5.69) $\mathbf{a}^{-1} = \left(\frac{1}{\mathbf{a}}\right) = \frac{\mathbf{a}}{\mathbf{a}^2}$

makes it possible to *divide* with a 1-vector in the same plane.

(5.70) $\mathbf{Ba}^{-1} = -\mathbf{a}^{-1}\mathbf{B}$

or rather multiplying by the inverse 1-vector from the right or left is anti-commuting.[222]

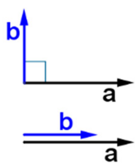
5.2.5.5. The Category a Bivector

We conclude the fundamental ***category*** for the conceptual bivector idea:
Bivectors may be the same or different. An individual bivector can be divided into several bivectors, and different bivectors can be combined into one bivector. A bivector ***quality*** we give by a ***direction*** unity-plane-segment bivector $\widehat{\mathbf{B}}$ by def. (5.65) applied to (5.66), hence

(5.71) $\widehat{\mathbf{B}}^2 = -\left|\widehat{\mathbf{B}}\right|^2 = -1$

The squared normalized ***quantity*** of a plane-segment ***direction*** $\widehat{\mathbf{B}}$ is then $-1 \in \mathbb{R}^1_{\text{pqg-2}}$.
The bivector ***quantity*** is simply performed by $\mathbf{B} = \beta\widehat{\mathbf{B}}$ for $\forall\,\beta \in \mathbb{R}^1_{\text{pqg-2}}$.
This factor β span a plane from $\widehat{\mathbf{B}}$.

Bivectors have existence or not by multiplication of 1-vectors in two important particular cases of possible existing 1-vectors:
Orthogonal 1-vectors anticommute $\mathbf{b}\cdot\mathbf{a} = 0 \Leftrightarrow \mathbf{ba} = -\mathbf{ab}$, $\boxed{\cos\theta = 0}$
Collinear 1-vectors commute $\mathbf{a}=\lambda\mathbf{b} \Leftrightarrow \mathbf{b}\wedge\mathbf{a} = 0 \Leftrightarrow \mathbf{ba}=\mathbf{ab}$, $\boxed{\sin\theta = 0}$
a 1-vector $\mathbf{a}$ is colinear with itself, so $\mathbf{a}\wedge\mathbf{a} = 0$, for $\forall\mathbf{a}$, as well $\mathbf{a}\cdot\mathbf{a} = \mathbf{a}^2$



Two *colinear* 1-vectors do not constitute a bivector.
Two *orthogonal* 1-vectors constitute a wedge product for a plane rectangular area.

[220] In Geometric Algebra (Clifford Algebra) the *pseudoscalars* are the highest ***grade*** elements in the ***primary quality grades*** that are necessary for the algebra. Euclidian *pseudoscalars* square to a negative scalar and commute with all even elements.
[221] A third 1-vector $\mathbf{c}$ will give impact $\mathbf{b}\wedge\mathbf{a}\wedge\mathbf{c} \neq 0$ when it is exterior to the plane $\mathbf{b}\wedge\mathbf{a}$. (see later below chapter 6).
[222] The division symbol $\frac{\mathbf{B}}{\mathbf{a}}$ makes no sense! But multiplication by the inverse 1-vector $\mathbf{a}^{-1} = \frac{1}{\mathbf{a}}$ from the right or the left is allowed.

5.2.6. The Orthonormal Bivector Object as a Unit for the Circular Rotation in a Plane

We look at the orthogonal unit vectors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ also called an orthonormal basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$ for a plane. – By definition, it applies a priori:
Orthogonal: $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 = 0$ and Normalised: $\boldsymbol{\sigma}_1{}^2 = \boldsymbol{\sigma}_2{}^2 = 1 \Rightarrow |\boldsymbol{\sigma}_1| = |\boldsymbol{\sigma}_2| = 1$.
From here we form a bivector for the plane that we call $\boxed{\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1}$.
Since $\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_1 = 0$, for this bivector we have

(5.72) $\boldsymbol{i} = \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1 = \frac{1}{2}(\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$

as just the product of the two orthonormal basis vectors for the plane and by antisymmetric permutation, we have the two orientations[223]

(5.73) $\boxed{\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1}$ with the commuted $\boxed{-\boldsymbol{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}$.

Which means that $\boldsymbol{i}$ is the special multivector constituted by one bivector.
According to $|\boldsymbol{\sigma}_1| = |\boldsymbol{\sigma}_2| = 1$ in (5.66) and $\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_1 = 0 \Rightarrow \sin^2\theta = 1$ in (5.62), together with (5.71) we have the auto product

(5.74) $\boldsymbol{ii} = \boldsymbol{i}^2 = -1$.

This leads to the normalized magnitude of $|\boldsymbol{i}| = |-\boldsymbol{i}| = 1$
Therefore, both $\boldsymbol{i}$ and $-\boldsymbol{i}$ are the two unitary bivectors.
The unit for the ***direction*** of a plane segment $\widehat{\mathbf{B}}$ has two eigenstates

(5.75) $\widehat{\mathbf{B}} = \pm\boldsymbol{i} = \pm 1\boldsymbol{i}$, in that $\widehat{\mathbf{B}}^2 = \boldsymbol{i}^2 = -|\boldsymbol{i}|^2 = -\left|\widehat{\mathbf{B}}\right|^2 = -1$

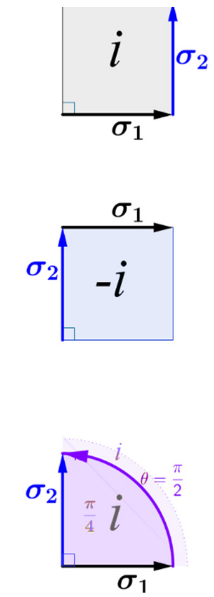


Figure 5.12 Unit 2-blade $\boldsymbol{i} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ or $-\boldsymbol{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ and $\pm\pi/2$ rotation objects.

We say that the unit-plane-segment ***direction*** $\widehat{\mathbf{B}}$ has two eigenvalues 1 and -1.
Compared with quantum mechanics we intuit $\boldsymbol{i}$ as a ***direction*** *operator* for a unit-area-segment.
Any arbitrary plane area $\beta=|\mathbf{B}| \geq 0$ provided by a bivector $\mathbf{B} = \beta\widehat{\mathbf{B}}$ ***quantity*** for a plane-segment ***pqg***-2 ***direction*** thus has two eigenstates $\mathbf{B}^+ = +\beta\boldsymbol{i}$ or $\mathbf{B}^- = -\beta\boldsymbol{i}$ and the ***quantitative*** eigenvalues $+\beta$ and $-\beta$ for each area. When you have an area, you should seriously consider its orientation and which of the two bivectors $\mathbf{B}$ or $-\mathbf{B}$ you use for intuition.

5.2.6.2. The Hodge Coordinate for the Pseudoscalar Span in the 𝔓 plane Concept

All bivector pseudoscalars in the plane 𝔓 idea are proportional to the basic unit bivector

(5.76) $\mathbf{B} = \beta\boldsymbol{i}$

For all $\forall\beta \in \mathbb{R}$ we have the Hodge[224] map: $\beta \to (*\beta) = \mathbf{B} = \beta\boldsymbol{i}$ for the plane idea.
This is a linear one-to-one map from the real numbers to the pseudoscalars of the ***directional primary quality of second grade*** (***pqg***-2) for the 𝔓 plane concept. These pseudoscalars represent the ***directional*** area ***quantity*** of a plane, where the negative parameter coordinates $\beta<0$ represent the retrograde area opposite orientated to a progressive area $\beta > 0$. ($\beta = 0$ represent every ***pqg***-0 point in 𝔓 without any ***direction***).

5.2.6.3. Operations with the Unit Bivector Pseudoscalar for a Plane

The operator $\boldsymbol{i}$ acts on the space concept 𝔖 and creates one plane ***direction***.
Implicitly $\boldsymbol{i} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$ is given by the two orthonormal geometric 1-vector-operators. First $\boldsymbol{\sigma}_1$ operates in space and sets a linear ***direction***, then $\boldsymbol{\sigma}_2$ operates perpendicular to $\boldsymbol{\sigma}_1$ through space and by that spans a plane ***direction*** through the plane unit segment $\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$.

[223] I am sorry to tell you that this book uses the reversed order of that first defined by David Hestenes in [6] and [5] (11).
It is essential for the intuition in this book that we use the sequential left operational order in vector multiplication like function operation $f\circ g = f(g) = fg$. Then the unit pseudoscalar bivector for the plane is $\boxed{\boldsymbol{i} \equiv \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1}$.
[224] The idea to call this a Hodge map of the form $\beta \to *\beta$ is taken from reference [35] .