5.2.5. The Outer Product of Geometric Vectors [213]

We look at the anti-symmetric part $\mathbf{a}\wedge\mathbf{b}$ by commutating the geometric product (5.60)

(5.60) $\quad \mathbf{ba} = \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\wedge\mathbf{b}$, as alternative to $\mathbf{ba} = \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\wedge\mathbf{a}$

We multiply these two equations (5.59) and (5.60) as a product[214] of two 2-multivectors

(5.61) $\quad \mathbf{abba} = (\mathbf{a}\cdot\mathbf{b})^2 - (\mathbf{a}\wedge\mathbf{b})^2$

As we defined above (5.42) $\mathbf{bb} = \mathbf{b}^2 = |\mathbf{b}|^2$ is a scalar, and[215] next $\mathbf{aa} = \mathbf{a}^2 = |\mathbf{a}|^2$ is a scalar, then $\mathbf{abba} = |\mathbf{a}|^2|\mathbf{b}|^2$, and from (5.49) we have the scalar $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \in \mathbb{R}$, therefore

(5.62) $$(\mathbf{a}\wedge\mathbf{b})^2 = (\mathbf{a}\cdot\mathbf{b})^2 - \mathbf{abba} = (|\mathbf{a}||\mathbf{b}|\cos\theta)^2 - |\mathbf{a}|^2|\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2(\cos^2\theta - 1) = -|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta \le 0.$$

This square of the outer product is negative and therefore $\mathbf{a}\wedge\mathbf{b}$ is not a scalar, but an anticommuting pseudoscalar in the plane, this idea is a subject in the ***pqg***-2 substance. Seen from outside the plane we call this type of object a **bivector**, as an outer product of two 1-vectors. This is when considered as a concept subject idea, also called for a **2-blade**.[216]
The letter expression for a bivector $\mathbf{A} = \mathbf{a}\wedge\mathbf{b}$ or $\mathbf{B} = \mathbf{b}\wedge\mathbf{a}$ is here named in large bold Latin letters.[217] For the magnitude of the bivector $\mathbf{B} = -\mathbf{A} = \mathbf{a}\wedge\mathbf{b}$ we get from (5.62) the area

(5.63) $\quad |\mathbf{B}| = |-\mathbf{A}| = |\mathbf{b}\wedge\mathbf{a}| = |\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\sin\theta| \ge 0, \in\mathbb{R}_+,\quad$ where $\theta = \sphericalangle(\mathbf{a},\mathbf{b}) \in \mathbb{R}$.

This is the formula for the parallelogram area spanned by the two 1-vectors **a**, **b** (Figure 5.20 below). The anti-symmetry in (5.58) gives the two orientations of the ***direction*** for the rotation idea as an area segment, that the bivector spans in the plane.

5.2.5.2. The Bivector Orientation as a Sequential Operation

The **Bivector** $\mathbf{B} = \mathbf{b}\wedge\mathbf{a}$ has a reverse-orientated $\mathbf{A} = -\mathbf{B} = \mathbf{a}\wedge\mathbf{b}$ in its rotation ***direction***.

Figure 5.10 The ***quality*** of the object **Bivector** $\mathbf{b}\wedge\mathbf{a}$ spans the area ***quantity*** magnitude $|\mathbf{b}\wedge\mathbf{a}|$.
The area spanned ***direction*** $\mathbf{B} = \mathbf{b}\wedge\mathbf{a}$ is here progressive orientated (contra-clockwise ↺) by the operation order **a**,**b**.
It has a reversed ordered **b**,**a** outer product $\mathbf{A} = -\mathbf{B} = \mathbf{a}\wedge\mathbf{b}$ orientated retrograde from **b** to **a** around origo O (clockwise ↻).
The bivector concept gives the area ***quality*** in a plane with two orientations $(-,+)$ of its ***quantity*** $\pm|\mathbf{b}\wedge\mathbf{a}|$.

Because of the anti-commutation rule (5.58) $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$ the operation order of the 1-vectors **a** and **b** have an impact. First, we let in our intuition the 1-vector **a** operate on the space substance, then the 1-vector **b** operates through ∧ on the impact of **a** and thus on space, and we have $\mathbf{B} = \mathbf{b}\wedge\,\mathbf{a}$. The vector operator $\mathbf{b}\wedge$ pulls the 1-vector **a** through a plane in space, creating a plane-segment $\mathbf{B} = \mathbf{b}\wedge\mathbf{a}$. – We read from left to right, but when we consider 1-vectors as operator the far right operates first then the next left, and so on successively sequential.[218]
Therefore, the orientation of the rotation ***direction*** for the plane segment is positively orientated from **a** to **b** around the origo O. This is opposite the reading orientation of $\mathbf{b}\wedge\mathbf{a}$, the operation sequence orientation of $\mathbf{b}\wedge(\mathbf{a}(\,))$ from the inner parenthesis with the successive left operation.

[213] These sections are made with great inspiration from the works of David Hestenes [33] and [10] etc.
[214] As full fills the same geometric algebra for products as described by the a priori rules in (5.38)-(5.41).
[215] We recall that scalars commute with vectors and vector products (5.41).
[216] The name 2-blade for a basis bivector is often used as part of a more general concept of k-blades $(a_k\wedge\ldots\wedge a_2\wedge a_1)$ as a subspace of dimension $k \le n$ basis of k 1-vectors $a_i \in V_n$ for i=1, ... k in a geometric algebraic space $\mathcal{G}(V_n)$ over vector space is V_n.
[217] Here bold capitals in a plane concept of a geometric space $\mathcal{G}(V_3)$. For $n>3$, $\mathcal{G}(V_n)$ we will use *non-bold italic letters* $A, B \cdots$.
[218] Refer to written functional principle $f\circ g = f(g) = fg \not\approx gf = g(f) = g\circ f$ for the left operation sequential order of functions.

In Figure 5.10, this is displayed as the progressive orientation ↺+ from 1-vector object **a** to **b** forming a representative subject bivector $\mathbf{B} = \mathbf{b}\wedge\mathbf{a}$ as a plane substance ***direction quality***.
The reverse-orientated bivector $\mathbf{A} = -\mathbf{B} = \mathbf{a}\wedge\mathbf{b}$ is shown as the operator $\mathbf{a}\wedge$ pulling 1-vector **b** through space and forming a plane segment $\mathbf{a}\wedge\mathbf{b}$ with retrograde orientation from 1-vector object **b** to **a** forming the subject $\mathbf{A} = -\mathbf{B} = -\mathbf{b}\wedge\mathbf{a}$ with *reversed* orientation ↻− of the area ***direction*** of $-\mathbf{B}$. In addition to the two rotation orientations of a plane segment area $|\mathbf{B}| = |-\mathbf{B}|$ provided by the opposite bivectors $\mathbf{a}\wedge\mathbf{b}$ and $\mathbf{b}\wedge\mathbf{a}$, the plane area segment has also a front and a back of a surface (paper\screen seen by an external observer). If the rotation ***direction*** as seen from the front is in a clockwise orientation, it is counterclockwise viewed from the back.
Consequently, only the sequential algebraic order of the 1-vector operations has an impact on the *reversion*. This is very important for the intuition of a bivector concept as a pure algebraic object definition $\mathbf{B} = \mathbf{b}\wedge\mathbf{a}$ of a geometric subject. The orientation ↻(−),↺(+) is here irrelevant since it depends on whether the outer observer is in front or at the back of the paper (f-b-rationale), like seeing the clock face at the front or from the back.
Later below for generalised multivectors, we will use † to mark the *reversed*, e.g., then $\mathbf{B}^\dagger = \mathbf{a}\wedge\mathbf{b}$.

5.2.5.3. Bivector *Quantity* and Form Structure *Quality* Direction

Just as the geometric a ***pqg***-1-vector line-segment object can be written as a scalar ***quantity*** α multiplied by a ***direction*** unit 1-vector $\hat{\mathbf{a}}$ representing the ***pqg***-1 ***quality***

(5.64) $\quad \mathbf{a} = \alpha\hat{\mathbf{a}} = |\mathbf{a}|\mathbf{u}$, where $\alpha = |\mathbf{a}| \in \mathbb{R}_+$, and $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \mathbf{u}$, that $|\hat{\mathbf{a}}| = |\mathbf{u}| = 1 \in \mathbb{R}_+$,

a bivector **B** is also written as a real scalar ***quantity*** β multiplied by a unit bivector $\widehat{\mathbf{B}}$ as a plane ***direction*** in space, representing the pure ***pqg***-2 ***quality***

(5.65) $\quad \mathbf{B} = \beta\widehat{\mathbf{B}} = |\mathbf{B}|\widehat{\mathbf{B}}$, where $\beta = |\mathbf{B}| \in \mathbb{R}_+$, and $\widehat{\mathbf{B}} = \frac{\mathbf{B}}{|\mathbf{B}|}$ that $|\widehat{\mathbf{B}}| = 1 \in \mathbb{R}_+$.

$\widehat{\mathbf{B}}$ is the unit for the ***direction*** of a plane segment as the ***primary quality of second grade*** (***pqg***-2).
Thus $\mathbf{B} = \beta\widehat{\mathbf{B}}$ is also a plane ***pqg***-2 ***quality***. The ***quantity*** of the bivector is its area magnitude $\beta = |\mathbf{B}| \in \mathbb{R}_{\text{pqg-2}}$ and the unit bivector has unit area magnitude $|\widehat{\mathbf{B}}| = 1 \in \mathbb{R}_+$, where $\widehat{\mathbf{B}}$ is the special unit ***pqg***-2 ***direction quality*** of a plane in the space concept $\mathfrak{G}$ in physics.
Negative values of the quotient scalar $\beta_- = (-\beta) < 0$ are allowed, e.g., the opposite orientation $\mathbf{A} = -\mathbf{B} = \beta_-\,\widehat{\mathbf{B}}$. This makes $\mathbf{B} = \beta\widehat{\mathbf{B}}$ a ***quantity*** of type $\forall\beta \in [\mathbb{R}_{\text{pqg-2}}]$ in the plane given by $\widehat{\mathbf{B}}$.
The bivector $\mathbf{B} = \beta\widehat{\mathbf{B}}$ is independent of the plane-segment shape in the plane space, a parallelogram rectangle, or circle etc. only the ***direction*** $\widehat{\mathbf{B}}$ and magnitude $\beta = |\mathbf{B}|$ of the area is preserved.
This is the preserved ***quantity*** joint with its ***quality*** independent of the amoeba shape change.

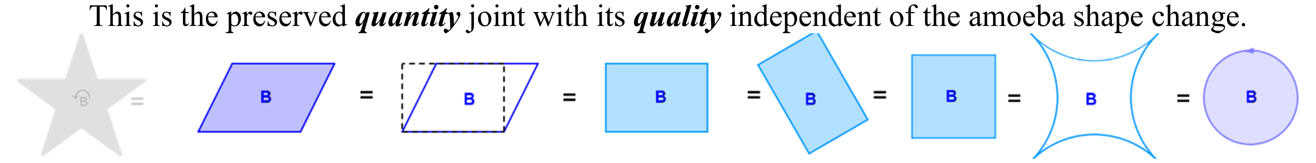

Figure 5.11 The **B** bivector does not have any specific geometrical shape but represents a plane segment with ***pqg***-2 ***direction***. Iconic the circular shape is preferable because it just highlights the plane symmetry and no linear ***pqg***-1 ***direction***. The bivector circle or the figure shape defines the exact plane of the surface it is drawn on as an object for the intuition of the plane concept $\mathfrak{P}$.

Do we have a 1-vector **a** in the plane for a bivector **B** we can find[219] a 1-vector **b** in the plane so $\mathbf{B} = \mathbf{b}\wedge\mathbf{a}$. This bivector **B** can with definition (5.59) be rewritten $\mathbf{B} = \mathbf{b}\wedge\mathbf{a} = \mathbf{ba} - \mathbf{b}\cdot\mathbf{a}$.
Here the scalar product can disappear $\mathbf{b}\cdot\mathbf{a} = 0$ according to (5.49) for the perpendicular 1-vector $\mathbf{b}\perp\mathbf{a}$, which performs a rectangle $\boxed{\mathbf{B}}$ that corresponds precisely to the plane segment **B**. (⊥ gives $\cos\theta = 0$ and $\sin\theta = 1$), whereby we due (5.58) and (5.62) apply

(5.66) $$\left.\begin{aligned}\mathbf{B} &= \mathbf{ba} = -\mathbf{ab} = \mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a},\\ \mathbf{B}^2 &= -|\mathbf{B}|^2 = -|\mathbf{b}|^2|\mathbf{a}|^2,\end{aligned}\right\}\quad \text{when } \mathbf{b}\perp\mathbf{a} \Leftrightarrow \mathbf{b}\cdot\mathbf{a} = 0, \text{ a rectangle}$$

[219] The linear combination for every 1-vector in the plane (5.23) enables this.

For quotation reference use: ISBN-13: 978-8797246931