whose real argument $\theta \in \mathbb{R}$ is the angular arc measure, as a $\mathbb{R}_{pqg\text{-}2}$ ***quantity***. The function value $\cos\theta$ is a real scalar ***quality pqg***-0 whose real $\mathbb{R}_{pqg\text{-}0}$ ***quantity*** is the ratio of the orthogonal ⊥ projection of the 1-vector $\mathbf{u}_2$ on $\mathbf{u}_1$ as a colinear (parallel ∥) part and the 1-vector $\mathbf{u}_1$ itself.
We form two new 1-vectors by dilation $\mathbf{a}=\alpha\mathbf{u}_1$ and $\mathbf{b}=\beta\mathbf{u}_2$ located in the same plane as shown in Figure 5.8. From this, we form a scalar product between the two vectors

(5.49) $$\mathbf{a}\cdot\mathbf{b} = \alpha\mathbf{u}_1\cdot\mathbf{b} = \alpha\mathbf{u}_1\cdot\beta\mathbf{u}_2 = \alpha\beta\,\mathbf{u}_1\cdot\mathbf{u}_2 = |\mathbf{a}||\mathbf{b}|\,\mathbf{u}_1\cdot\mathbf{u}_2 = |\mathbf{a}||\mathbf{b}|\cos\theta, \text{ where } \theta = \measuredangle(\mathbf{a},\mathbf{b}).$$

Note that the $\mathbf{b}$ projection[210] of the 1-vector $\mathbf{a}$ has a magnitude $\mathbf{u}_1\cdot\mathbf{b} = \beta\mathbf{u}_1\cdot\mathbf{u}_2$,
to be scaled by the magnitude $\alpha=|\mathbf{a}|$ to the scalar-product $\alpha\beta\cos\theta$. And the symmetry dictates also that the projection of $\mathbf{a}$ on the 1-vector $\mathbf{b}$ has the magnitude $\mathbf{u}_2\cdot\mathbf{a} = \alpha\mathbf{u}_2\cdot\mathbf{u}_1$ scaled by the magnitude $\beta=|\mathbf{b}|$ of $\mathbf{b}$ to give the same scalar-product, thus commutative

(5.50) $$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = \beta\alpha\cos\theta.$$

This scalar product indicates the magnitude of the projection of the one 1-vector on the other, multiplied by the magnitude of this other 1-vector, and vice versa, shown in Figure 5.9 $\mathbf{a}\cdot\mathbf{b}=\alpha\mathbf{u}_1\cdot\mathbf{b}=\beta\mathbf{u}_2\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{a}$
The commutative algebra symmetry is expressed (5.49).
The scalar-product ***quantity*** $\mathbb{R}_{pqg\text{-}0}$ which we intuit as an object of ***grade***-0. That would say an object without geometric extension, but merely just the symmetrical co-linear scalar projection ratio between the two angled 1-vectors (co-sinus) multiplied by their magnitudes.

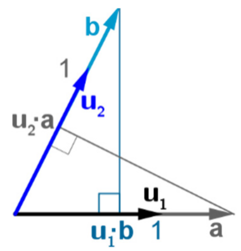

Figure 5.9 $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ is commutative.

In the case of a right angle in (5.49) ⊥ $\sim\measuredangle(\mathbf{a},\mathbf{b}) = \theta = \frac{\pi}{2}$, then $\cos\theta=0$ and thus $\mathbf{a}\cdot\mathbf{b} = 0$ as well as $\mathbf{u}_1\cdot\mathbf{u}_2 = 0 \in \mathbb{R}_{pqg\text{-}0}$. Whenever $\mathbf{u}_1\cdot\mathbf{u}_2 = 0$ we call the two 1-vectors $\mathbf{u}_1, \mathbf{u}_2$ orthogonal, hence the 1-vectors $\mathbf{a} = \alpha\mathbf{u}_1$ and $\mathbf{b} = \beta\mathbf{u}_2$ will be orthogonal too.
The inner product $\mathbf{a}\cdot\mathbf{b}=0$ means that the two orthogonal 1-vectors are independent in their ***pqg***-1 ***quality*** expressed as **no** ***pqg***-0 scalar cosine ***quantity*** in their plane relation.
For geometric 1-vectors,[211] the inner product is a real scalar $\mathbb{R}$.

The inner product commute and is therefore symmetrical between the 1-vectors $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

The fact, that the scalar product commute between 1-vectors in a plane means, that it is independent of the space outside the plane, therefore the designation as *inner product*, in accordance with (5.45)-(5.47).

For an angle with $\measuredangle(\mathbf{a},\mathbf{b}) = \theta = 0$, we have $\cos\theta = 1 \in \mathbb{R}$ and the two 1-vectors are co-linear, and the scalar product is the same as the product of the two magnitudes $|\mathbf{a}||\mathbf{b}|$.
This is comparable to § 4.4.4.1, where $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 \Leftrightarrow |\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{(\mathbf{a})^2}$ is the metric of 1-vectors through this quadratic form.

For 1-vectors in the Euclidean plane, the auto scalar product is never negative $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2\geq 0$.

[210] For the phenomenological understanding of the orthogonal projection of the 1-vector **b** on **a** it should be noted that the projection is a 1-vector $P_{\mathbf{a}}(\mathbf{b}) = (\mathbf{u}_1\cdot\mathbf{b})\mathbf{u}_1 = (\hat{\mathbf{a}}\cdot\mathbf{b})\hat{\mathbf{a}} = \left(\frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$, in the ***direction*** $\mathbf{u}_1 = \hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$.

[211] In a generalised vector space, the inner product of two vectors of the same grades is also a scalar. As an example, the use of complex vector space $\psi,\varphi \in (V,\mathbb{C})$, where we often write the inner product as $\langle\psi,\varphi\rangle = \langle\psi|\varphi\rangle = \psi\cdot\varphi$

Research on the a priori of Physics
Geometric Critique of Pure Mathematical Reasoning
Jens Erfurt Andresen
Edition 2, © 2020-22

5.2.3. The Inner Product Algebra

We prescribe the commutative algebraic conditions for the inner product in a Euclidean plane

(5.51) $$\begin{cases} \text{(i)} & \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}, & \text{The commutative law.} \\ \text{(ii)} & \mathbf{a}\cdot(\lambda\mathbf{b}) = \lambda(\mathbf{a}\cdot\mathbf{b}),\ \ \lambda\in\mathbb{R}, & \text{the scalar multiplicative law.} \\ \text{(iii)} & \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}, & \text{The distributive low over addition.} \\ \text{(iv)} & \mathbf{a}\cdot\mathbf{a} > 0 \ \vee\ \mathbf{a} = \mathbf{0} \Rightarrow \mathbf{a}\cdot\mathbf{a} = 0, & \text{Euclidean metric norm.} \end{cases}$$

From the difference between two 1-vectors we can deduct the inner product (see Figure 5.7)

$$\mathbf{d} = \mathbf{a} - \mathbf{b} \quad \Rightarrow$$
$$\mathbf{d}\cdot\mathbf{d} = |\mathbf{d}|^2 = |\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot(\mathbf{a}-\mathbf{b})-\mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2+|\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}$$

(5.52) $$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \tfrac{1}{2}(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2)$$

And from the sum of two 1-vectors $\mathbf{c}=\mathbf{a}+\mathbf{b}$ we obtained the inner scalar product $\mathbf{a}\cdot\mathbf{b}$ as

$$\mathbf{c} = \mathbf{a}+\mathbf{b} \quad \Rightarrow$$
$$\mathbf{c}\cdot\mathbf{c} = |\mathbf{c}|^2 = |\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot(\mathbf{a}+\mathbf{b})+\mathbf{b}\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2+|\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b}$$

(5.53) $$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \tfrac{1}{2}(|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}|^2 - |\mathbf{b}|^2)$$

From the quadratic form (5.46) we see that the inner product is

(5.54) $$\mathbf{a}\cdot\mathbf{b} \coloneqq \tfrac{1}{2}(\mathbf{ab}+\mathbf{ba}) = \tfrac{1}{2}((\mathbf{a}+\mathbf{b})^2-\mathbf{a}^2-\mathbf{b}^2) = \tfrac{1}{2}(\mathbf{a}^2+\mathbf{b}^2-(\mathbf{a}-\mathbf{b})^2) = \tfrac{1}{4}((\mathbf{a}+\mathbf{b})^2-(\mathbf{a}-\mathbf{b})^2)$$

Two geometric 1-vectors together form an inner scalar product. From the ordinary vector geometry, we repeat the simple formula (5.49)

(5.55) $$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \in \mathbb{R}, \quad \text{where } \theta = \measuredangle(\mathbf{a},\mathbf{b}) \in \mathbb{R}.$$

The scalar product forms a pure scalar ***quantity*** measure $\mathbb{R}_{pqg\text{-}0}$ for the symmetric colinear internal relations between the two 1-vectors. In general, the scalar is a measure of a colinear internal dependency in a physical ***entity*** expressed between two mutual related 1-vectors.

In addition to this scalar measure, the anti-symmetry between the two geometric vectors from (5.44) forms a plane concept as a ***primary quality of second grade*** (***pqg***-2).
This plane substance we intuit as an objective surface that we see from its outside, therefor the anti-symmetry of the product is called an *outer* ***quality***.

5.2.4. The Geometric Product

We return to the general product of geometric vectors **a** and **b** from (5.44)

(5.56) $$\mathbf{ab} = \tfrac{1}{2}(\mathbf{ab}+\mathbf{ba}) + \tfrac{1}{2}(\mathbf{ab}-\mathbf{ba}).$$

We have seen (5.46)-(5.54) that the first part is a symmetrical commuting *inner product*

(5.57) $$\mathbf{a}\cdot\mathbf{b} = \tfrac{1}{2}(\mathbf{ab}+\mathbf{ba}) \qquad \text{(is a real scalar).}$$

This symmetrical *inner product* has also been called the *interior product*.
For the last antisymmetric part, we will write with a wedge angle icon ∧ between the two vectors

(5.58) $$\mathbf{a}\wedge\mathbf{b} = \tfrac{1}{2}(\mathbf{ab}-\mathbf{ba}) = -\tfrac{1}{2}(\mathbf{ba}-\mathbf{ab}) = -\mathbf{b}\wedge\mathbf{a} \qquad \text{(is a bivector).}$$

This part of the product is called the anti-commuting *outer* product (or *the exterior product*).
In this way, the geometric product of 1-vectors can be written as

(5.59) $$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}$$

This is an example of a so-called 2-*multivector*,[212] or just a 2-vector.

[212] 2-multivector, 2 stands for the simple product polynomials of two 1-vectors and scalars, e.g., just **ab** or $\gamma\mathbf{ab}+\beta\mathbf{c}+\alpha$.