5.2.1.5. Clifford Algebra

We look at a vector space V over a field $\mathbb{K}$ that is equipped with a quadratic form Q

$$\mathbf{v}^2 = Q(\mathbf{v})\epsilon_A \quad \text{for } \forall \mathbf{v} \in V, \tag{5.36}$$

where $\mathbf{v}^2 = \mathbf{vv}$ is an algebra product, and ϵ_A is the multiplicative identity in the algebra.
This is called a Clifford algebra and we designate it by $C\ell(V, \mathrm{Q})$.
For geometric 1-vectors the quadratic forms are a real function, that is,
the linear vector space $V_{\mathrm{pqg\text{-}1}}$ is over the real numbers $[\mathbb{R}_{\mathrm{pqg\text{-}1}}]$

$$\mathbf{v}^2 = Q(\mathbf{v})\,\epsilon_A = \epsilon_A|\mathbf{v}|^2 \in \mathbb{R} \quad \text{for } \forall \mathbf{v} \in V \tag{5.37}$$

In practice,[208] the multiplicative signatures are $\epsilon_A = 1, 0, -1$.
We define the *magnitude* as $|\mathbf{v}| = |\mathbf{v}^2|^{½}$, or in general $|A| = |A^2|^{½}$
In Euclidean space $\mathbb{R}^n$, we apply $\epsilon_A = 1 \in \mathbb{R}$, just as in Euclidean geometry, i.e., $\mathbf{v}^2 = |\mathbf{v}|^2 \in \mathbb{R}_+$,
where the norm $|\mathbf{v}| = \sqrt{|\mathbf{v}^2|}$ gives the magnitude of $\mathbf{v}$. It is well known, that the natural geometric plane and the natural 3-dimensional space of physics is Euclidean with the signature $\epsilon_A = 1$.

5.2.1.6. The Combined Linear Space

The plane subjects, intuit as objects $(\mathbf{v}, \mathbf{u})$, $\mathbf{v} \neq \lambda\mathbf{u}$ for $\forall\lambda \in \mathbb{R}$, are given by two ***quantities***:
First the magnitude as a pure scalar, e.g.: $|(\mathbf{v}, \mathbf{u})| \in \mathbb{R}_{\mathrm{pqg\text{-}0}}$, and second the angle $\mathbb{R}_{\mathrm{pqg\text{-}2}}$ (illustrated in Figure 5.3 and Figure 5.4), or the perpendicular ***direction*** (as the tangent illustrated in Figure 5.6). These can be combined in two separate ways:

1. as a direct sum $\mathbb{R}\oplus\mathbb{R} \sim \mathbb{R}_{\mathrm{pqg\text{-}1}}\oplus\mathbb{R}_{\mathrm{pqg\text{-}1}} = \mathbb{R}^2_{\mathrm{pqg\text{-}1}} \rightarrow \mathbb{R}^1_{\mathrm{pqg\text{-}1}}$, as (5.23) and
2. as a direct product $\mathbb{R}_{\mathrm{pqg\text{-}1}}\otimes\mathbb{R}_{\mathrm{pqg\text{-}1}} = \mathbb{R}_{\mathrm{pqg\text{-}0}}\oplus\mathbb{R}_{\mathrm{pqg\text{-}2}} \sim \mathbb{R}_{\mathrm{pqg\text{-}0}}\otimes\left(\mathbb{R}_{\mathrm{pqg\text{-}2}}(\mathbb{R}_{\mathrm{pqg\text{-}0}})\right)$, as below (5.59).

The first combination is the addition of two linearly independent 1-vectors. In this, the angle is transcendental implicitly given by necessity, possibly by a Cartesian right angle.
The second combination is formed by the product of two 1-vectors.
In the tradition, we usually have for the complex scalars an associated abstract plane. Further in this tradition, we usually have $\mathbb{R}\oplus\mathbb{R} = \mathbb{R}^2 \leftrightarrow \mathbb{C}$ with the complex scalars for the plane.
The first combination is then the addition of a complex real part Re and imaginary parts Im.
The second combination is a product of a real number $\rho \in \mathbb{R}_{\mathrm{pqg\text{-}0}}$, and a complex unitary function $e^{i\varphi} \in \mathbb{C}$, of another real number $\varphi \in \mathbb{R}_{\mathrm{pqg\text{-}2}}$. – We will examine this geometric further below. –
As is well known, and previously described in Section 4.1.3 - § 4.1.3.3, we know in our intuition that the geometric complex vector space $(V_m, \mathbb{C}) \sim (V, \mathbb{C}^m,)$ is synonymous with a real linear space $(V_n, \mathbb{R}) \sim (V, \mathbb{R}^{2m})$, $n = 2m$, therefore, we intuit a geometric algebra of finite dimensional real vector space $(V_n, \mathbb{R}) \sim (V, \mathbb{R}^n)$, with a quadratic form $Q: V \rightarrow \mathbb{R}$, which is a so-called real field $\mathbb{R}$ Clifford algebra $C\ell(V, Q)$. – A complex field structure will camouflage the geometric foundation.

5.2.2. The Geometric Algebra with Direct Product

In addition to the additive algebra for vectors, we ethical expand the algebra with rules for the multiplication of vectors. For the product of such vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ from the vector space V we shall apply the fundamental multiplication rules

$$\mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c}, \quad \text{the associative law for products,} \tag{5.38}$$

$$\mathbf{a}(\mathbf{b}+\mathbf{c}) = \mathbf{ab} + \mathbf{ac}, \quad \text{the product is left distributive for addition,} \tag{5.39}$$

$$(\mathbf{b}+\mathbf{c})\mathbf{a} = \mathbf{ba} + \mathbf{ca}, \quad \text{the product is right distributive for addition,} \tag{5.40}$$

$$\mathbf{a}\lambda = \lambda\mathbf{a} \Rightarrow \mathbf{ab}\lambda = \mathbf{a}\lambda\mathbf{b} = \lambda\mathbf{ab}, \ \lambda \in \mathbb{R}, \quad \text{the commutative law for scalar multiplication} \tag{5.41}$$

$$\mathbf{a}^2 = \mathbf{a}\cdot\mathbf{a} = \mathbf{aa} = \epsilon_A|\mathbf{a}|^2 = \pm|\mathbf{a}|^2 \in \mathbb{R}, \quad \text{scalar metric (contraction). (If } \epsilon_A = 0 \Rightarrow \mathbf{a}^2 = 0) \tag{5.42}$$

[208] The multiplicative signature $\epsilon_A = 1, 0, -1$ has its cause in permutations. An example of a negative signature is the complex numbers where $(i)^2 = -1$ and something similar we see in the Minkowski metric.

Vectors that meet this multiplicative algebra we call *geometric vectors*, e.g. **ab** in which the product of geometric vectors is called a *geometric product*. We expand this concept with sums of geometric products which we call **multi-vector** of the form $M = \mathbf{ab} + \mathbf{cd}$ ... etc.
Generally, we use the typography in *Italic* capitals for geometric[207] multi-vectors e.g., A, M, P ... and further $M = \alpha AB + \beta CD + \gamma E\mathrm{f} + \delta \mathrm{g}H + \cdots$. (Exceptions are 1-vectors **a** and bivectors **B**.)
For multi-vectors, we apply the geometric algebra (5.38)-(5.42), which is not commutative by multiplication, but in all meets the same additive algebra as vectors (5.11)-(5.21).
This algebra is called a (Clifford) *geometric algebra* named $\mathcal{G}(V)$ over the vector space V.
Comment, for the associative product $\mathbf{b}(\mathbf{ba}) = \mathbf{b}^2\mathbf{a}$, we get a scalar $\mathbf{b}^2$ multiplied by a vector **a**.
The squared 1-vector objects $\mathbf{a}^2, \mathbf{b}^2 \ldots \in \mathbb{R}$ are pure scalar ***quantities*** $\mathbb{R}_{\mathrm{pqg\text{-}0}}$, see § 4.4.4.1.
If the vector **a** is not colinear (not parallel) with **b** the product **ba** is not just such a scalar.
Using the scalar $\lambda = -1$ from (5.41) as a factor for the addition to get subtraction, we can judge a priori, that the auto-subtraction annihilates (disappears)

$$\mathbf{ba} + (-1)\mathbf{ba} = \ \mathbf{ba} - \mathbf{ba} = 0. \tag{5.43}$$

It helps us to dissolve product **ab** by inserting the reverse product **ba**.
By this, we split and expand the geometric product in a *symmetrical* and an *anti-symmetrical* part

$$\underbrace{\mathbf{ab}}_{product} = \underbrace{\tfrac{1}{2}(\mathbf{ab} + \mathbf{ba})}_{symmetric\ inner} + \underbrace{\tfrac{1}{2}(\mathbf{ab} - \mathbf{ba})}_{anti\text{-}symmetric\ outer}. \qquad \text{Reason is} \quad \mathbf{ab} = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ab} + \mathbf{ba} - \mathbf{ba}). \tag{5.44}$$

- The last term in brackets has an impact by an external exchange of the two vectors **a** and **b**.
- The first term in brackets is in its symmetry independent by external exchange $\mathbf{ab} \leftrightarrow \mathbf{ba}$, hence, a commutation does not influence the inner part of the product.

We are setting $B = \mathbf{ab}$ and $B^{\sim} = \mathbf{ba}$ as well as $B^+ = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$, $B^- = \frac{1}{2}(\mathbf{ab} - \mathbf{ba})$, just as in the bilinear form (5.28) (5.31) and (5.32), we get the split (5.33) $B = B^+ + B^-$

5.2.2.2. The Inner Symmetric Product of Geometric Vectors

Two geometric 1-vectors are generating a plane. We, therefore, turn to the link between two vectors. Similar to the bilinear form we are now introducing the general vector product **ab** for $\forall \mathbf{a}, \mathbf{b} \in V$ and claim the above algebra $\mathcal{G}(V) \simeq C\ell(V, \mathrm{Q})$ for such geometric product of vectors. We start from the quadratic form (5.37) for the sum of two vectors

$$Q(\mathbf{a}+\mathbf{b}) = (\mathbf{a}+\mathbf{b})^2 = (\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b}) = \mathbf{aa} + \mathbf{ab} + \mathbf{ba} + \mathbf{bb} = \ \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{ab} + \mathbf{ba} \tag{5.45}$$

Using the symmetrical bilinear form (5.31), we define the *inner product*

$$\mathbf{a}\cdot\mathbf{b} \equiv \langle\mathbf{a}, \mathbf{b}\rangle \coloneqq \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) = \tfrac{1}{2}\left((\mathbf{a}+\mathbf{b})^2 - \mathbf{a}^2 - \mathbf{b}^2\right) = \tfrac{1}{2}\left(Q(\mathbf{a}+\mathbf{b}) - Q(\mathbf{a}) - Q(\mathbf{b})\right), \tag{5.46}$$

as a symmetrical bilinear form $\mathbf{a}\cdot\mathbf{b} \sim B^+(\mathbf{a}, \mathbf{b})$ associated with the quadratic form $\boldsymbol{Q}$.
The inner product commute according to the symmetrical definition

$$\mathbf{a}\cdot\mathbf{b} \equiv \langle\mathbf{a}, \mathbf{b}\rangle \coloneqq \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) = \tfrac{1}{2}(\mathbf{ba} + \mathbf{ab}) \ = \langle\mathbf{b}, \mathbf{a}\rangle = \mathbf{b}\cdot\mathbf{a} \tag{5.47}$$

5.2.2.3. The Scalar Product

In § 5.1.1.5 as shown in Figure 5.2, we have two 1-vectors $\mathbf{e}_1 = \overrightarrow{\mathrm{OA}}$ and $\mathbf{u}_2$ each generating half lines that intersect as an angle from their origo O.
We rename $\mathbf{e}_1$ to $\mathbf{u}_1 = \mathbf{e}_1$, so the u's stand for free unitary ***directions***, that is $|\mathbf{u}_2| = |\mathbf{u}_1| = 1$.
The two 1-vectors $\mathbf{u}_1 \neq \mathbf{u}_2$ span a plane from O.
Historically we have created the real scalar function[209] cosine

$$\mathbf{u}_1\cdot\mathbf{u}_2 = \cos\theta \quad \in \mathbb{R}, \tag{5.48}$$

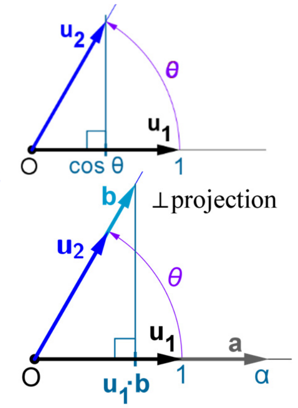

Figure 5.8 The scalar product.

[209] Co-sinus means together by the arc or under the arc (Greek: sinus). In practice, the projection is down on the ground line.