5.2. The Plane Geometric Algebra

5.2.1. Addition of 1-vectors in the Plane

Now that we have an intuition of the plane angle concept, we will look at the addition of two 1-vectors that form an angle to each other.
As an object for us, the addition is displayed for intuition in Figure 5.7.
First the sum

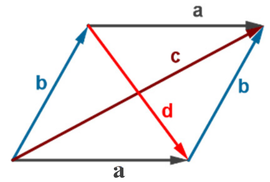

Figure 5.7 Vector sum and difference.

(5.9) $\mathbf{c} = \mathbf{a} + \mathbf{b}$,

then the additive difference between the 1-vectors

(5.10) $\mathbf{d} = \mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}$. Where alternatively $\mathbf{a} = \mathbf{b} + \mathbf{d}$

The subtraction of the 1-vector $\mathbf{b}$ is the addition of the inverse orientation obtained by the scalar factor (-1).
The fact that three 1-vectors added together, do not necessarily exist in the same plane, we need to successively select a new plane for an angle every time we add one 1-vector to a sum of 1-vectors. Generally, geometric 1-vectors satisfy the general additive algebra of vector spaces.

5.2.1.2. The Additive Algebra for Vector Spaces of Geometric Substance

In section 4.1.1.1 we defined the general rules (4.1)-(4.11) for an arbitrary linear vector space over a scalar field $\mathbb{K}$. For the natural geometric vector space for physics, we will limit ourselves to using only the real numbers $\mathbb{R}$ as the scalar field for the vector space $(V,\mathbb{R}) \sim V$ with the additive identical neutral element $\mathbf{0} \in V$ called the zero-vector.
We use bold lowercase Latin letters to denote the physical (objective) geometric 1-vectors and rewrite the additive algebra of this linear vector space $(V,\mathbb{R}) \sim V$
For arbitrary geometric elements $\mathbf{a},\mathbf{b},\mathbf{c} \in V$ we apply the following algebraic rules:

(5.11) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$, the associative law for addition.

(5.12) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, the commutative law for addition.

(5.13) $\exists \mathbf{0} \in \mathrm{V}\colon\ \mathbf{a} + \mathbf{0} = \mathbf{a}$ for $\forall \mathbf{a} \in \mathrm{V}$, the identical element for addition, the zero-vector $\mathbf{0}$.

(5.14) $\forall \mathbf{a} \in \mathrm{V},\ \exists {-\mathbf{a}} \in \mathrm{V} \Rightarrow \mathbf{a} + (-\mathbf{a}) = \mathbf{0}$, where $-\mathbf{a}$ is the additive inverse orientation to $\mathbf{a}$.

(5.15) $\alpha(\beta\mathbf{b}) = (\alpha\beta)\mathbf{b}$ for $\alpha,\beta \in \mathbb{R}$, the associative scalar field multiplication.

(5.16) $1\mathbf{a} = \mathbf{a}$, where $1 \in \mathbb{R}$, the multiplicative identical neutral scalar $1 \in \mathbb{R}$.

(5.17) $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$, distributive scalar multiplication for vector addition.

(5.18) $(\alpha + \beta)\mathbf{c} = \alpha\mathbf{c} + \beta\mathbf{c}$, distributive scalar multiplication for scalar addition.

(5.19) $\lambda\mathbf{a} = \mathbf{a}\lambda$, $\lambda \in \mathbb{R}$ commutative multiplication by the scalar field.

(5.20) Subtraction of a vector is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

(5.21) Division of a vector with a scalar is defined as $\dfrac{\mathbf{a}}{\alpha} = \dfrac{1}{\alpha}\mathbf{a}$

This general linear algebra for linear spaces also applies to geometric 1-vectors. Multiplication with real scalars of 1-vectors is a ***primary quality of first grade*** (***pqg***-1) where the physical ***quantity*** $[\mathbb{R}^1_{+\mathrm{pqg}\text{-}1}]$ is obtained by multiplication with a scalar $\lambda \in \mathbb{R}$.
The addition of two 1-vectors $\mathbf{a}$ and $\mathbf{b}$ in the same plane provides a new 1-vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$ also of the ***quality pqg***-1, even when they form an angle with each other and hence are linearly independent.
It is left to the reader by the intuition of Figure 5.7 to confirm the a priori judgment

(5.22) $\alpha\mathbf{a} + \beta\mathbf{b} = 0 \Rightarrow \alpha = \beta = 0$.

The plane angle between the two linearly independent 1-vectors $\mathbf{a}$ and $\mathbf{b}$ forms a mutual relationship forming a ***primary quality of second grade*** (***pqg***-2).

5.2.1.3. The Linear Span of the Geometrical Plane from 1-vectors

Given a 1-vector $\mathbf{a}$ and another 1-vector $\mathbf{b}$ with an objective proper mutual ***pqg***-2 angle, forming a plane $\gamma_{\mathbf{ab}}$, we can write any 1-vector $\mathbf{y}$ in the plane as a linear combination

(5.23) $\mathbf{y} = \alpha\mathbf{a} + \beta\mathbf{b}$, where $\alpha,\beta \in \mathbb{R}$, and $\mathbf{y}$ is ***pqg***-1-vector.

This form is called a linear representation of the plane spanned by the two 1-vectors $\mathbf{a}$ and $\mathbf{b}$.
Although the substance of the plane $\gamma_{\mathbf{ab}}$ requires a ***pqg***-2 angular ***quality***, all objects $\mathbf{y}$ in the linear span (5.23) are 1-vector subjects of ***quality pqg***-1, even they lay in the plane $\gamma_{\mathbf{ab}}$.
We just get $\mathbf{y} \in \mathbb{R}^2_{\mathbf{a},\mathbf{b}} = \mathbb{R}^1_{\mathbf{a}} \oplus \mathbb{R}^1_{\mathbf{b}} \sim \mathbb{R}_{\mathrm{pqg}\text{-}1} \oplus \mathbb{R}_{\mathrm{pqg}\text{-}1} = \mathbb{R}^2_{\mathrm{pqg}\text{-}1} \rightarrow \mathbb{R}^1_{\mathrm{pqg}\text{-}1}$. This indicates strongly that 1-vector span production is not a full adequate presentation for the plane concept which substance ***quality*** requires ***pqg***-2 angles which corresponds to circular arcs. See Figure 5.1 etc.
We, therefore, introduced a new substance of the space concept $\mathfrak{G}$, which we call the concept of *multi-vectors*, where we generally describe the elements in uppercase *Italic letters* $A, B, \cdots M, \cdots Z$.[207]

5.2.1.4. Bilinear Forms

For a general vector space V, we may apply bilinear mapping $B\colon V \times V \rightarrow \mathbb{K}$, linearly in each argument. We look at the bilinear form for vectors $\mathbf{u},\mathbf{v},\mathbf{w} \in V$ from the vector space

(5.24) $B(\mathbf{u}+\mathbf{v},\mathbf{w}) = B(\mathbf{u},\mathbf{w}) + B(\mathbf{v},\mathbf{w})$

(5.25) $B(\mathbf{u},\mathbf{v}+\mathbf{w}) = B(\mathbf{u},\mathbf{v}) + B(\mathbf{u},\mathbf{w})$

(5.26) $B(\lambda\mathbf{u},\mathbf{v}) = B(\mathbf{u},\lambda\mathbf{v}) = \lambda B(\mathbf{u},\mathbf{v})$, $\lambda \in \mathbb{K}$

We call the bilinear form B non-degenerated if we can apply the rules

(5.27) $B(\mathbf{v},\mathbf{w}) = 0$ for $\forall \mathbf{w} \in V \Rightarrow \mathbf{v} = \mathbf{0}$ or $B(\mathbf{v},\mathbf{w}) = 0$ for $\forall \mathbf{v} \in V \Rightarrow \mathbf{w} = \mathbf{0}$

We define the transpose (reverse) bilinear form

(5.28) $B^{\sim}(\mathbf{v},\mathbf{w}) = B(\mathbf{w},\mathbf{v})$

The bilinear form is called symmetrical, when

(5.29) $B(\mathbf{v},\mathbf{w}) = B(\mathbf{w},\mathbf{v})$

and conversely, the bilinear form is called antisymmetric, when

(5.30) $B(\mathbf{v},\mathbf{w}) = -B(\mathbf{w},\mathbf{v})$

For any bilinear form B, we can construct the symmetrical part of the bilinear form

(5.31) $B^{+} = \frac{1}{2}(B + B^{\sim})$, $\quad B^{+}(\mathbf{v},\mathbf{w}) = \frac{1}{2}\big(B(\mathbf{v},\mathbf{w}) + B(\mathbf{w},\mathbf{v})\big) = B^{+}(\mathbf{w},\mathbf{v})$

and the anti-symmetric part of the bilinear form

(5.32) $B^{-} = \frac{1}{2}(B - B^{\sim})$, $\quad B^{-}(\mathbf{v},\mathbf{w}) = \frac{1}{2}\big(B(\mathbf{v},\mathbf{w}) - B(\mathbf{w},\mathbf{v})\big) = -B^{-}(\mathbf{w},\mathbf{v})$

We hereby have the split expansion of the bilinear form

(5.33) $B(\mathbf{v},\mathbf{w}) = \frac{1}{2}\big(B(\mathbf{v},\mathbf{w}) + B(\mathbf{w},\mathbf{v})\big) + \frac{1}{2}\big(B(\mathbf{v},\mathbf{w}) - B(\mathbf{w},\mathbf{v})\big) = B^{+}(\mathbf{v},\mathbf{w}) + B^{-}(\mathbf{w},\mathbf{v})$

From the bilinear form $B\colon V \times V \rightarrow \mathbb{K}$, we associate the quadratic form

(5.34) $Q\colon V \rightarrow \mathbb{K} : \mathbf{v} \mapsto B(\mathbf{v},\mathbf{v})$

Using the quadratic form $Q(\mathbf{v})$ in (5.31) and (5.32) we see convergence with the symmetrical form and the exclusion of the anti-symmetric part. $\mathbf{v} \rightarrow Q(\mathbf{v}) = B(\mathbf{v},\mathbf{v})$.
Generally, for a quadratic form, we from (5.26) apply

(5.35) $Q(\lambda\mathbf{v}) = \lambda^2 Q(\mathbf{v})$, for $\forall \mathbf{v} \in V$, $\forall \lambda \in \mathbb{K}$

[207] Even though the little bold Latin alphabet $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \cdots \mathbf{z}$, is used for geometric 1-vectors concerned with the *multi-vector* concept. And for the concept of Euclidean *bivector objects*, we will use uppercase bold Latin letters $\mathbf{A},\mathbf{B},\cdots$. In all other cases $A, B, \cdots$.