4.1.5. Qualitative Substance of a Vector Space

Above, we have demonstrated that the concept of a linear space as a vector space $(V, \mathbb{K})$ depends on the ability to find a linearly independent basis set which can create the span of a vector space. To make the vector space $(V, \mathbb{K})$ represent an ***entity*** in physics the vector space substance must have a ***quality*** which is already carried by the basis set. All the individual basis vectors must have an equal ***quality*** of the same type, so that together they can form a vector space, in addition, they must have different type of ***qualities*** that makes them *distinguishable* and linearly independent.

This common equal ***quality*** must then have a corresponding scalar ***quantity***, which couple them to the multiplicative scalar unit $1 \in \mathbb{K}$ so that it is possible to produce a linear combination of the basis set over $\mathbb{K}$, which then hands the spanned vector space V. The basis set as an object has an overall ***quality grade*** transferred to the entire vector space $(V, \mathbb{K})$, which substance in this way inherits this ***quality***. This overall ***quality grade*** consists of the common ***quality*** for all the basis vectors and the individual ***qualities*** that make these basis vectors *distinguishable*. For relevance, they must be linear independent.

Here we must remember our ethics, that the idea of vector spaces with basis sets is a human art as thought construction (Res Cogitans) and therefore have no existence outside the idea.

But the thought objects as basis vectors and basis sets may have reference to ***entities*** as objects in physics (res extensa) of natural space and in some cases only abstract subject ***entities*** to make a model to understand a substance of physics. (E.g., space-time itself)

4.1.5.1. Linear Relationship of Geometry

Descartes related numeric symbols with geometric line segments to let their numerical value represent the magnitude of the segment. He expanded the idea of rational numeric relations to a geometry continuum and created the possibility for the development of scalars where we attach the real numbers to the geometric line concept. (E.g., see [10]p5→.)

This has allowed the use of vector spaces to represent geometry, and we have developed additive algebra to describe the linear relation in the geometry of physics.

To elaborate on this, we will look further into this geometry for physics.

4.1.5.2. The Geometry

We can describe space by taking some basic elements of Euclid's geometry, and then adding some knowledge that we otherwise have about natural space.

I start with the breakdown in dimensions.

Historically it is just the Euclidean plane geometry that we are cognitively familiar with in knowing the concept of a plane represented by the paper, board, or tabula, where Points, Lines, Circles, Triangles, Squares, etc. are well-known abstract concepts.

We say that the plane geometry is 2-dimensional 2D.

These two dimensions we know from the (x, y)-coordinate system for the plane concept.

When it comes to the 3-dimensional natural space in physics, it is more complicated.

The first to discover the congruence problem for three dimensions was Immanuel Kant [11] who pointed out that Leibniz ignored the symmetry problem. - Leibniz claim, that **form** of space we can fully define through **magnitude** relationships, which he called '*analysis situs* '. –

Kant introduced an additional factor: ***direction*** for the spatial extension.

The symmetry problem Kant discovered; we will certainly not be able to solve only by a linear additive algebra for a vector space.[174]

– There must be something more. –

[174] *analysis situs* from Leibniz is following the linear additive algebra, e.g. represented by a simple Cartesian coordinate system $\mathbb{R}^3$, or 3D. – but –

4.2. The Geometric Space $\mathfrak{G}$

Euclidean geometry consists of rules for Points A, Lines ℓ, Circles ⨀, Triangles △, squares □, angles ∠, and other figures.[175] Together these concepts do exist in our intuition as a theory of natural space. This represents the idea of a concept:

The natural geometric space $\mathfrak{G}$ of physics is transcendental for us as it is the tool by which we recognize the outside world. The word "geo" (earth) represents the world around us and the word 'metry' is our intuition tool. Geometry – is well known for more than three thousand years, but the abstract transcendental character of the geometric space as a substance has been taboo.

Now we give it the name $\mathfrak{G}$, viewing it by intuition not only as a space, – but a completely geometric universe, with which we view the world. We assume that $\mathfrak{G}$ has a structure which we call the substance of the *total natural space* $\mathfrak{G}$ for physics.

The a priori synthetic judgments we can pass on the geometric spaces $\mathfrak{G}$ are as follows:

(4.52) $$\forall A \in \mathfrak{G} \;\Rightarrow\; \ell \subset \mathfrak{G},\quad \odot \subset \mathfrak{G},\quad \triangle \subset \mathfrak{G},\quad \square \subset \mathfrak{G},\quad \forall\,\text{figur} \subset \mathfrak{G},\ \text{and}\ \ \forall\angle \in \mathfrak{G}.$$

Euclid's Elements together with Hilbert's attempt at a complete axiomatic geometry has taught us[176] that it is only possible to have a logical geometric system for plane geometry $\mathfrak{P}$.

Still, we must deal with dimensions beyond the plane geometrical foundation.

In space geometry (stereo-metrics of solids, etc.), we traditionally operate with a concept of **dimensions nD that we introduce in the following table:**

nD	Concept of dimensionality	Concept definition		Synthetic judgment	Noumenon (Plato ideal)	Concept of grades Extra characteristics	Scalar fields and Pseudoscalar	
0D	Concept of a Point	$\mathfrak{X}$	$A \in \mathfrak{X}$	$A \in \mathfrak{G}$	$A \in \mathfrak{G}$	$\mathfrak{X} \subseteq \mathfrak{G}$	$\{0\} \subset \mathbb{R}$	
1D	Concept of a Line	$\mathfrak{L}$	$\ell \in \mathfrak{L}$	$A \in \ell \subset \mathfrak{G}$	$\ell \in^{?} \mathfrak{G}$	$\vec{\mathfrak{L}} = \mathfrak{L} \oplus \mapsto$ line ***direction***	$\vec{\mathbb{R}} \sim \mathbb{R}$	→
2D	Concept of a Plan	$\mathfrak{P}$	$\gamma \in \mathfrak{P}$	$A \in \gamma \subset \mathfrak{G}$	$\gamma \in^{?} \mathfrak{G}$	$\mathfrak{P}^{\circlearrowleft} = \mathfrak{P} \otimes \circlearrowleft$ rotation and angular ***direction***	$\mathbb{C}^{\circlearrowleft} \nsim \mathbb{C} \ni i$ Minkowski $\mathcal{B}$-plan	$\boldsymbol{i}$ $\mathcal{B}$
3D	Concept of a space (stereo metric)	$\mathfrak{Z}$	$\mathfrak{Z}$-space	$A \in \mathfrak{Z} \subset \mathfrak{G}$	$\mathfrak{Z} \in^{?} \mathfrak{G}$	$\mathfrak{Z}^{\circlearrowright} = \mathfrak{Z} \otimes \circlearrowright$ spin, chiral ***direction***	Pseudo³scalar (Pauli algebra)	$\boldsymbol{i}$
4D	Development and relations (relative)	$\mathfrak{D}$	$\mathfrak{D}$-space	$X \in D \subset \mathfrak{G}$	$\mathfrak{D} \in^{?} \mathfrak{G}$	Causal helicity ***direction***	Pseudo⁴scalar (STA-algebra)	i
∞	infinity	?	Eternal	Religion !	Transcendental, A priori foundation, A ¬Ω		0↔∞	∞

We make additional synthetic judgments $\ell \subset \gamma \subset \mathfrak{Z} \subseteq \mathfrak{G}$

In the same plane, may be different lines $\exists \ell_1, \ell_2 \subset \gamma\!: \;\; \ell_1 \neq \ell_2$

In the same space, may be different planes $\exists \gamma_1, \gamma_2 \subset \mathfrak{Z}\!: \;\; \gamma_1 \neq \gamma_2$

We know immediately that lines ℓ and planes γ, together with spatial local[177] structures $\circledast \subset \mathfrak{Z}$ are parts of the geometric space $\mathfrak{G}$. But is it also the case that they are elements of $\mathfrak{G}$?

Since the space $\mathfrak{G}$ is the form of intuition by which we recognize the world, we must assume that it has existed in physics, although the space $\mathfrak{G}$ is transcendental to our recognition. (Spinoza, Kant) $\mathfrak{G}$ is non-perceptive. We know that the lines ℓ , planes γ and space $\mathfrak{Z}$ are platonic ideas that we can understand. Immanuel Kant called this Noumenon, – the intelligible.

It is therefore obvious that they can only be elements of an idealised space $\forall \ell, \forall \gamma \in \text{Ideal}(\mathfrak{G}) \rightarrow \mathfrak{G}$, which we by intuition related to the universal geometric space $\mathfrak{G}$, with which we recognize physics. –

[175] We assumed that the reader has basic knowledge of the Euclidean geometry.

[176] Supported by Gödel's incompleteness proof of the concept of numbers. And the later theories of chaos.

[177] The local structure $\circledast$ is finite as opposed to ℓ, γ and $\mathfrak{Z}$. The synthetic a priori judgment is for a locality $\circledast \neq \mathfrak{Z}$.