(4.43) $s_\infty(\tau) = \sum_{\forall n\in\mathbb{Z}} \alpha_n e^{i2\pi n\nu\tau} \quad \in \mathbb{C} \rightarrowtail \mathbb{C}^\infty_\mathbb{Z} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}^1_n, \quad$ for $\nu\in\mathbb{R},\ \tau\in\mathbb{R}$

The coefficients in the linear combination we write as an integral over one whole period of an arbitrary periodic function with the period $\frac{1}{\nu}$ as follows

(4.44) $\alpha_n = \nu \int_{\tau_0}^{\tau_0+\frac{1}{\nu}} s_\infty(\tau)\, e^{-i2\pi n\nu\tau} d\tau \in\mathbb{C}, \quad$ for $\nu\in\mathbb{R},\ \forall\tau\in\mathbb{R},\ \forall n\in\mathbb{Z}$.

Now we not only let $N\to\infty$ but also $\nu\to 0$, and thus the period $\frac{1}{\nu}\to\infty$ and hereby leave the periodicity of the function $s_\infty(\tau)$. With the writing $\omega_n = 2\pi n\nu$, and $\Delta\omega = 2\pi\nu\to d\omega$, and by this, we rewrite the integral in (4.44) as a function $\tilde{q}$ of ω_n, where

(4.45) $\alpha_n = \nu\, \tilde{q}(2\pi n\nu) = \frac{\Delta\omega}{2\pi}\tilde{q}(\omega_n)$.

Hence

(4.46) $s(\tau) = \frac{1}{2\pi}\sum_{\forall n\in\mathbb{Z}} \tilde{q}(\omega_n)\, e^{i\omega_n\tau}\Delta\omega \quad \in\mathbb{C} \rightarrowtail \bigoplus_{n\in\mathbb{Z}} \mathbb{C}^1_n = \mathbb{C}^\infty_\mathbb{Z}, \quad$ for $\tau\in\mathbb{R},\ \omega_n\in\mathbb{R}$.

We rename $s(\tau) \underset{\Delta\omega\to 0}{\longrightarrow} q(\tau)$ and thus $\omega_n\to\omega$ interpret by intuition as a continuous ***spectrum***.
It is a requirement that the functions $q(\tau)$ and $\tilde{q}(\omega)$ are integrable. – Thus, we get:

4.1.4.2. The Vector Space of Fourier Integrals

The numerable series (4.46) is expanded over a real continuum of complex basis functions $\hat{u}^*_\omega(\tau) = e^{i\omega\tau} \in\mathbb{C}^1_\omega$, where $\omega\in\mathbb{R}$, which indicates the basic functions that have the real argument $\tau\in\mathbb{R}$, and the real basis index $\omega\in\mathbb{R}$.
Then we have the linear spaces of integrable functions that we call an ***inverse Fourier integral***

(4.47) $\boxed{q(\tau) = \int_{-\infty}^{\infty} \tilde{q}(\omega)\cdot e^{i\omega\tau}\, d\omega = \int_{\omega\in\mathbb{R}} d\omega\tilde{q}(\omega)e^{i\omega\tau}} \quad \in\mathbb{C} \rightarrowtail \mathbb{C}^\infty_\mathbb{R}$. (1.80)

The integral we interpret by intuition as a linear combination spanned over the basis set

(4.48) $\left\{ e^{i\omega\tau} \in\mathbb{C}^1_\omega \subset \bigoplus_{\omega\in\mathbb{R}} \mathbb{C}^1_\omega = \mathbb{C}^\infty_\mathbb{R} \,\middle|\, \tau\in\vec{\mathbb{R}},\ \forall\omega\in\mathbb{R} \right\}$,

spanning the linear form (4.47) wherein the complex scalars $d\omega\,\tilde{q}(\omega)\in\mathbb{C}$ constitute the factors in the linear additive integral, at a specific local time $\tau\in\mathbb{R}$, as the *inverse Fourier vector space* $\mathbb{C}^\infty_\mathbb{R} = \text{span}\{ e^{i\omega\tau} \in\mathbb{C}^1_\omega \,|\, \tau, \forall\omega\in\mathbb{R}\}$, with $\dim(\mathbb{C}^\infty_\mathbb{R}) = \infty$, spanned of complex functions
In conjunction with this, we look at the *Fourier vector space*
$\mathbb{C}^\infty_\mathbb{R} = \text{span}\{ e^{-i\omega\tau} \in\mathbb{C}^1_\tau \,|\, \omega, \forall\tau\in\mathbb{R}\}$, of complex functions of a real argument $\omega\in\mathbb{R}$ over the basic functions of the same type, namely the ***quality*** of complex oscillating functions.
We then achieve a vector space $\mathbb{C}^\infty_\mathbb{R}$ feature of functions $\mathbb{R}\to\mathbb{C}$: $\tilde{q}(\omega)\in\mathbb{C}$ that we call the continuous ***spectrum*** of oscillators, which we find in the ***Fourier integral***

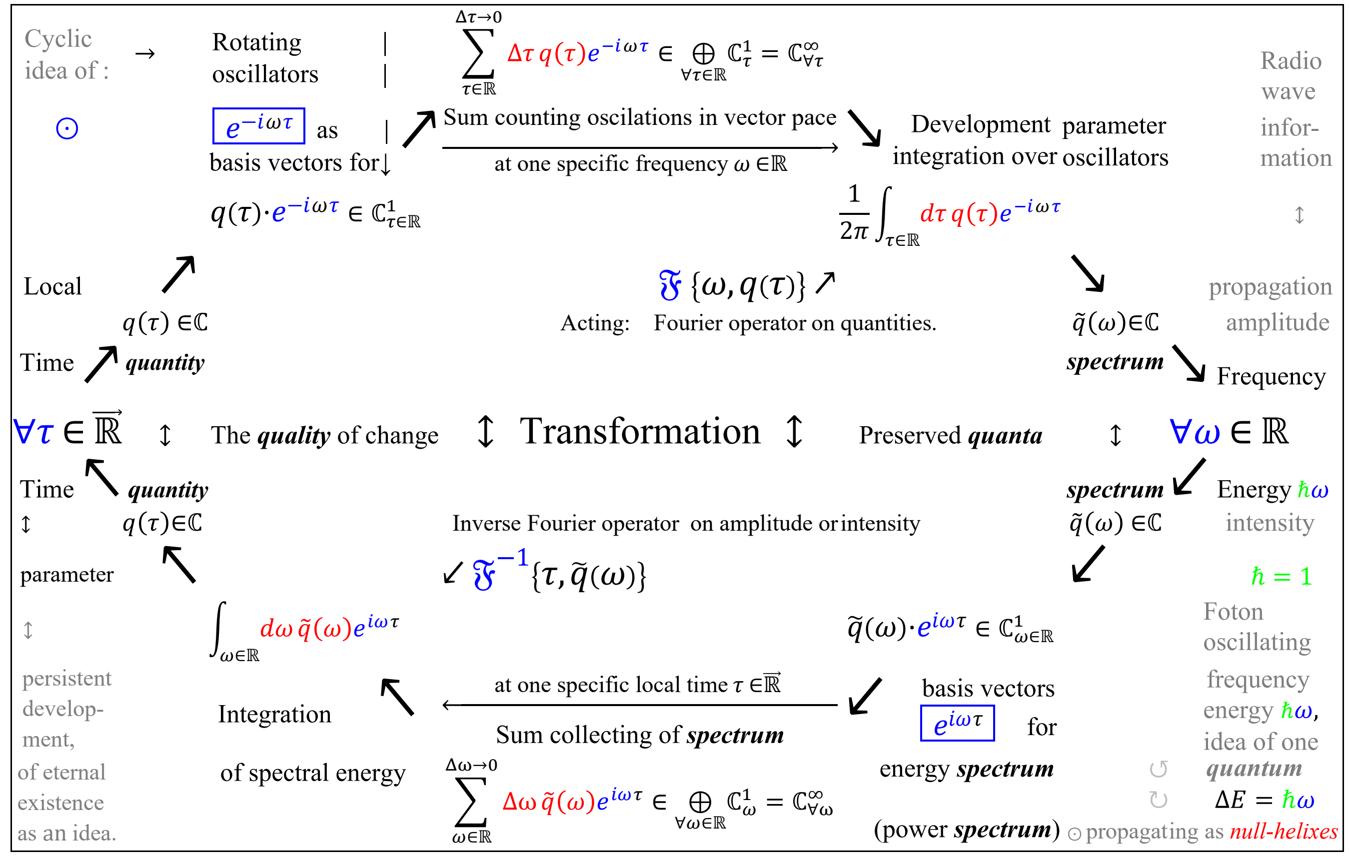
(4.49) $\boxed{\tilde{q}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(\tau)\cdot e^{-i\omega\tau}\, d\tau = \frac{1}{2\pi}\int_{\tau\in\mathbb{R}} d\tau q(\tau)e^{-i\omega\tau}} \quad \in\mathbb{C} \rightarrowtail \mathbb{C}^\infty_\mathbb{R}$. (1.81)

Here we interpret by intuition the integral as a linear span over a basis set of oscillations

(4.50) $\left\{ e^{-i\omega\tau} \in\mathbb{C}^1_\tau \subset \bigoplus_{\tau\in\mathbb{R}} \mathbb{C}^1_\tau = \mathbb{C}^\infty_\mathbb{R} \,\middle|\, \omega\in\mathbb{R},\ \forall\tau\in\vec{\mathbb{R}} \right\}$,

at a specific local frequency $\omega\in\mathbb{R}$ for (4.49) wherein the complex scalar factors consist of $d\tau q(\tau)\in\mathbb{C}$ in the linear additive integral.
The common arguments are referred to as $\tau,\omega\in\mathbb{R}$ for the two typical functions $q(\tau),\ \tilde{q}(\omega)\ \in\mathbb{C}$, and the basis functions as $\boxed{e^{i\omega\tau}}$, $\boxed{e^{-i\omega\tau}}\in\mathbb{C}$.
In physics, we intuit the basis vectors of a ***spectrum*** as oscillators with a transversal plane angular frequency energy $\omega\in\mathbb{R}$ and a time development parameter $\tau\in\mathbb{R}$. Therefore,

(4.51) we list the two dual ***Fourier transforms*** in a cyclic scheme : →



We are lucky that all function values of both basis vector sets $e^{i\omega\tau}$, $e^{-i\omega\tau}$ $\in\mathbb{C}$ are complex scalars like the scalars of the vector space, hence the integral values are complex scalars too.
Thus, we have the information development (time) parameter dependent ***quantity*** $q(\tau)$ is dual inverse through the Fourier integral operation on the energy ***spectrum*** $\tilde{q}(\omega)$ as shown in (4.47) and (4.49). See also Fourier transforms in Section 1.7.7.
Anyway, in the intuition of physics, the development of a scalar ***quantity*** $q(\tau)\in\mathbb{C}$ of an ***entity*** has the ***quality*** to be measured by counts relative to an infinite basis set of timing oscillator clocks $\{\hat{u}_\tau(\omega) = e^{-i\omega\tau} \in\mathbb{C}^\infty_\mathbb{R} \,|\, \forall\tau\in\vec{\mathbb{R}}\}$ from where we can span a linear space containing the weight factors $\tilde{q}(\omega)\in\mathbb{C}$ called a scalar ***spectrum*** over angular frequency energies $\omega\in\mathbb{R}$.
These spectral energy quantities have themselves a quality given by the infinite dual conjugated basis set of oscillations $\{\hat{u}^*_\omega(\tau) = e^{i\omega\tau} \in\mathbb{C}^\infty_\omega \,|\, \forall\omega\in\mathbb{R}\}$ from which we can span linear space that contains the possible developing quantity $q(\tau)\in\mathbb{C}$ for the entity. The reader may note for each $\omega\in\mathbb{R}$, the two dual oscillator basis sets are mutual independent even if contained in the same geometrical plane of their rotations in that they are reversal orientated to each other $\pm\omega$.

The philosophy of this epistemology expressed in the scheme above is that all time parameter values $\forall\tau\in\mathbb{R}$ are available simultaneously, called *the eternal time concept*. This is the mandatory prerequisite to say that each specific ***spectral*** oscillator frequency $\omega\in\mathbb{R}$ is constantly preserved (photon energy $\hbar\omega$). Classical expressed as of conservation of energy (eternally constant).
Opposite to fix a specific event time point $\tau = t\in\mathbb{R}$ demand all ***spectral*** frequencies $\forall\omega\in\mathbb{R}$, $-\infty\leq\omega\leq\infty$ be momentane available. That simply tells us to have all oscillator frequency energies present (called the ultraviolet catastrophe).
The impossibility of eternity and the absolute now presence is the cause of ***Quantum Mechanics***.
This dual complementarity is just the a priori foundation of what we call Heisenberg uncertainty.
To determine an absolute now requires infinite energy, and eternity requires no energy at all, where nothing happens. The synthetic judgment is $\Delta\tau\cdot\omega \gtrsim 1 \quad\leftrightarrow\quad \Delta\tau\cdot\Delta E \gtrsim \hbar$.