$\mathbb{C}_1^1 = \mathrm{span}\{1_\odot^\circlearrowleft\} = \{z = \psi 1_\odot^\circlearrowleft \,|\, \psi \in \mathbb{C}\}$, with the dimension $\dim(\mathbb{C}_1^1) = \dim(1_\odot^\circlearrowleft) = 1$,
as a representative of the concept of the complex plane.[171]

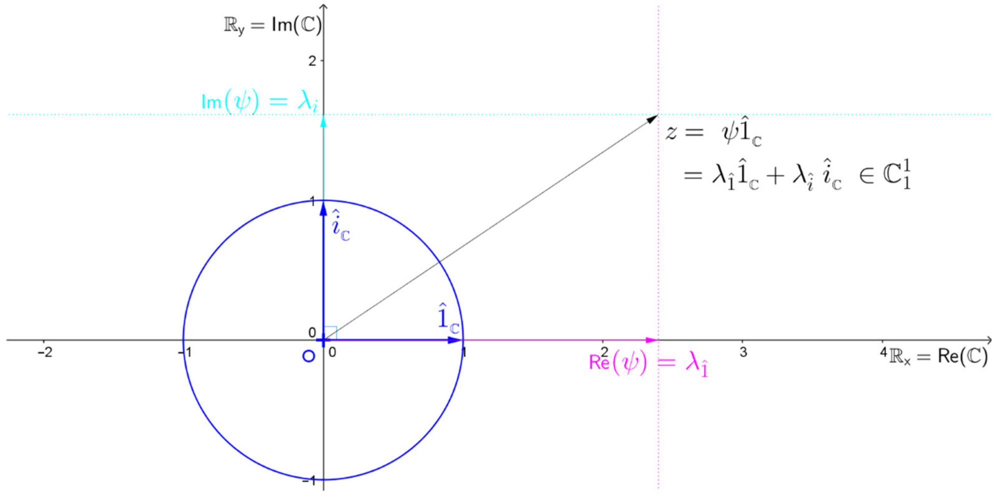

Figure 4.3 The complex unit circle is an example of one basis vector $1_\odot^\circlearrowleft$, which produces the complex plane.

4.1.3.2. The Complex Scalar

We look at scalars $\mathbb{C}$ themselves and its linear combinations $\psi = \psi 1 \in \mathbb{C}$ from the neutral element $1 \in \mathbb{C}$ for multiplication. We know that the complex numbers themselves have no extension, nor the neutral 1. Therefore, we must judge $Dim(\mathbb{C}) = 0$, – in physics!

4.1.3.3. A Multi-dimensional Complex Vector Space

We write a multidimensional complex space as $\mathbb{C}^n$, combined as $\mathbb{C}^n = \underbrace{\mathbb{C}_1^1 \oplus \ldots \mathbb{C}_j^1 \oplus \ldots \mathbb{C}_n^1}_{n}$.

We have an isometry with the real vector space $\mathbb{R}^{2n} \leftrightarrow \mathbb{C}^n$. I cannot formulate a direct intuitive graphic basis for $\mathbb{C}^n$, that deviates from what already identifies the real space $\mathbb{R}^{2n}$ for intuition. We must wait for chapter 6 to get a richer understanding of the higher dimensional complex rotational structure of objects in physics by Geometric Algebra.

[171] The low indices 1 in $\mathbb{C}_1^1$ indicate that it is the first complex plane object in a system of several planes $\ldots\mathbb{C}_j^1\ldots$ (or fields).

4.1.4. Vector Spaces of Infinite Dimensions

We can form vector spaces of functions by a scalar linear combination from basis functions.
Here we only look at complex functions with one real argument of type:

(4.35) $f: \mathbb{R} \to \mathbb{C},\ \lambda \to f(\lambda) \in \mathbb{C}$ for $\forall \lambda \in \mathbb{R}$

As we have seen, it is necessary to find a basis set to make an intuition of a vector space.
Here we will as the foundation use complex periodic basis-functions of the type[172]

(4.36) $\hat{u}_\tau(\omega) = e^{-i\omega\tau} \in \mathbb{C}_\tau^1$, over $\omega \in \mathbb{R}$,

where $\tau \in \mathbb{R}$ is the parameter that arguments the functions of $\omega \in \mathbb{R}$ as basis vectors, and τ will specify indexing of the infinite basis set of complex oscillating functions $\hat{u}_\tau(\omega)$ to our intuition. And as well we have their complementary conjugated

(4.37) $\hat{u}_\omega^*(\tau) = e^{i\omega\tau} \in \mathbb{C}_\omega^1$, over $\tau \in \mathbb{R}$

where $\omega \in \mathbb{R}$ is the parameter that arguments the functions of $\tau \in \mathbb{R}$ as basis vectors, and now ω will specify indexing of the infinite basis set of complex development functions $\hat{u}_\omega^*(\tau)$ to our intuition. Both these two sets of basic functions are linearly independent through randomly picked indices,[173] e.g., $\omega_j \neq \omega_k$

(4.38) $\alpha_j e^{-i\omega_j\tau} + \alpha_k e^{-i\omega_k\tau} = 0 \quad \Rightarrow \quad \alpha_j = 0,\ \alpha_k = 0$ for $\omega_j \neq \omega_k$,

because as we always apply $e^{i\omega\tau} \neq 0$, then by multiplying $e^{i\omega_k\tau}$ into (4.38) we get

(4.39) $\alpha_j e^{-i(\omega_j-\omega_k)\tau} + \alpha_k e^{-i0} = 0 \ \Rightarrow\ \alpha_j e^{-i(\omega_j-\omega_k)\tau} = -\alpha_k$ for $\forall \tau \in \mathbb{R}$,

Here $\alpha_j e^{-i(\omega_j-\omega_k)\tau}$ is constant $(-\alpha_k)$ for all arguments τ, which can only be the case when $\omega_j = \omega_k$ thereby $\alpha_j = -\alpha_k$, or $\alpha_j = 0$ thereby $\alpha_k = 0$, that is (4.38).
– Just the same with the complementary conjugate, e.g., $\tau_j \neq \tau_k$.
First, we look at a specific finite example, a vector space $(P_N, \mathbb{C})$ of periodic functions $s_N: \mathbb{R} \longrightarrow \mathbb{C}$ with a particular fundamental frequency $\nu \in \mathbb{R}$ with the period $\frac{1}{\nu}$, so that

(4.40) $s_N(\tau) = s_N\left(\tau + \frac{m}{\nu}\right) \in \mathbb{C}$ for $\forall \tau \in \mathbb{R},\ \forall m \in \mathbb{Z}$.

We choose the indexing n for the basic functions of $\tau \in \mathbb{R}$ indices parameterised with $\omega_n = 2\pi n\nu$.
Here the first basis vector is $e^{i2\pi\nu\tau} \in \mathbb{C}_1^1$, and then we constitute 'overtones' basis vectors $e^{i2\pi n\nu\tau}$ as the basis set of periodic functions

(4.41) $\left\{ e^{i2\pi n\nu\tau} \in \mathbb{C}_n^1 \subset \bigoplus_{n=-N}^{N} \mathbb{C}_n^1 = \mathbb{C}_\mathbb{Z}^{2N+1} \,\middle|\, n = -N \ldots -1,0,1 \ldots N \right\}$.

We then form a linear combination to what we call a ***Fourier series***

(4.42) $s_N(\tau) = \sum_{n=-N}^{N} \alpha_n e^{i2\pi n\nu\tau} \in \mathbb{C} \rightarrowtail \mathbb{C}_\mathbb{Z}^{2N+1} \leftrightarrow P_N$, for $\nu, \tau \in \mathbb{R}$, where $\alpha_{-n} = \alpha_n^*$

The function value is a complex scalar $s_N(\tau) \in \mathbb{C}$, while the functions $(s_N: \mathbb{R} \longrightarrow \mathbb{C}) \in \mathbb{C}_\mathbb{Z}^{2N+1}$ constitutes a vector space P_N having the dimension $\dim(P_N) = 2N+1$. Therefore, we have the writing form $s_N(\tau) \in \mathbb{C} \rightarrowtail \mathbb{C}_\mathbb{Z}^{2N+1}$ of the scalar which has its cause in this vector space.

We let $N \to \infty$ and thus get an infinite dimensional vector space $P^\infty \leftrightarrow \mathbb{C}_\mathbb{Z}^\infty = \bigoplus_{n=-\infty}^{\infty} \mathbb{C}_n^1$,

and thus, the infinite Fourier series for a periodic function $s_\infty(\tau) = s_\infty\left(\tau + \frac{m}{\nu}\right) \in \mathbb{C}, \forall m \in \mathbb{Z}$

[172] The designation term $\hat{u}_\tau(\omega)$ express that the indices τ represent the *basis choice of a function* of the argument parameter ω. complimented with $\hat{u}_\omega^*(\tau)$ express that the indices ω represent the *basis choice of function* of the argument parameter τ.
[173] For practice intuition we consider numerable index, pure mathematical reals are allowed. for background see § 1.7.8.2