The vector $a \in \mathbb{R}^2_{xy} = \mathbb{R}^1_x \oplus \mathbb{R}^1_y$ is thus represented by the two coordinate dimensions.
The vectors $x = \lambda_x \hat{1}_x \in \mathbb{R}^1_x$ and $y = \lambda_y \hat{1}_y \in \mathbb{R}^1_y$ also belong to $\mathbb{R}^2_{xy}$, as $\mathbb{R}^1_x \subset \mathbb{R}^2_{xy}$ and $\mathbb{R}^1_y \subset \mathbb{R}^2_{xy}$.
from the basis set $\{\hat{1}_x, \hat{1}_y\} \in \mathbb{R}^2_{xy}$, we form the vector space $\mathbb{R}^2_{xy} \sim (\mathbb{R}^2_{xy}, \mathbb{R})$ by the linear form

(4.27) $$a = \lambda_x \hat{1}_x + \lambda_y \hat{1}_y \in \mathbb{R}^2_{xy} \quad \text{for } \lambda_x, \lambda_y \in \mathbb{R}$$

over the real scalars and by that produce the real plane $\mathbb{R}^2_{xy} \sim \{ a = \lambda_x \hat{1}_x + \lambda_y \hat{1}_y \mid \forall \lambda_x, \lambda_y \in \mathbb{R}\}$.
If we remove the Cartesian x,y indexing in $\mathbb{R}^2_{xy}$, we lose the claim $\hat{1}_x \perp \hat{1}_y$ and we can use free geometric basis vectors e.g., $\{\hat{1}_1, \hat{1}_2\}$, (where $\hat{1}_1 \angle \hat{1}_2$) and the linear form span a plane

(4.28) $$\mathbb{R}^2 \sim \operatorname{span}\{\hat{1}_1, \hat{1}_2\} = \{x = \lambda_1 \hat{1}_1 + \lambda_2 \hat{1}_2 \mid \forall \lambda_1, \lambda_2 \in \mathbb{R}\}, \qquad \dim(\mathbb{R}^2) = 2.$$

The n dimensional real vector space $V_n \sim \mathbb{R}^n \sim (\mathbb{R}^n, \mathbb{R}) \sim (V_n, \mathbb{R})$ we can from a basis set $\hat{1}_1, \dots \hat{1}_n$ span ; $\mathbb{R}^n = \mathbb{R}^1_1 \oplus \dots \oplus \mathbb{R}^1_n = \operatorname{span}\{\hat{1}_1, \dots \hat{1}_n\}$ by a linear combination

(4.29) $$x = \lambda_1 \hat{1}_1 + \dots \lambda_n \hat{1}_n, \quad x \in \mathbb{R}^n, \quad \hat{1}_i \in \mathbb{R}^1_i \quad \text{and} \quad \lambda_i \in \mathbb{R} \quad \text{for } i = 1, \dots n, \quad \text{hence } \dim(\mathbb{R}^n) = n.$$

It is here left to the reader to consider the situation in three dimensions,
$\dim(\mathbb{R}^3) = 3$; $\mathbb{R}^3 = \mathbb{R}^3_{xyz} = \mathbb{R}^1_x \oplus \mathbb{R}^1_y \oplus \mathbb{R}^1_z$, for a linear form, that spans $V_3 \sim (\mathbb{R}^3, \mathbb{R})$ from a linearly independent basis set $\{\hat{1}_x, \hat{1}_y, \hat{1}_z\}$, and a scalar set $(\lambda_x, \lambda_y, \lambda_z)$.

4.1.3. The Vector Space of Complex Numbers

As with the real numbers $\mathbb{R}$, we look at the complex numbers $\mathbb{C}$ and form the corresponding vector space $\mathbb{C}^1 \sim (\mathbb{C}^1, \mathbb{C})$ over the complex number scalars $\mathbb{C}$.
Again, it is the multiplying neutral scalar $1 \in \mathbb{C}$, that must form the natural basis for $\mathbb{C}^1$
But the scalar $1 \in \mathbb{C}$ has as for the real numbers no extension $\overrightarrow{[1,1]} = 0$.
The real interval $\overrightarrow{[0,1]} \subset \mathbb{R}$ spans only the real numbers $\vec{\mathbb{R}}$. But a unit circle of a plane with a radius vector $\hat{1}_{\mathbb{C}} \in \mathbb{C}^1$ will be a graphic object as an intuition option for a basis.
Another vector $\hat{\imath}_{\mathbb{C}}$ perpendicular to $\hat{1}_{\mathbb{C}}$, $\hat{\imath}_{\mathbb{C}} \perp \hat{1}_{\mathbb{C}}$ has further defined the unit circle plane in a complex symbiosis. For this unit circle, we choose e.g. $\hat{1}_{\mathbb{C}} \leftarrow \hat{1}_x$, $\hat{\imath}_{\mathbb{C}} \leftarrow \hat{1}_y$ from the Cartesian plane where the linear form $a = \lambda_x \hat{1}_x + \lambda_y \hat{1}_y \in \mathbb{R}^2_{xy}$ for $\lambda_x, \lambda_y \in \mathbb{R}$ span the plane.
as above. Our intuition of a progressive orientated ***direction*** of rotation of this unit circle $\odot$ plane complex $1^{\circlearrowleft}_{\odot} \sim \hat{\imath}_{\mathbb{C}} \perp \hat{1}_{\mathbb{C}} \leftarrow \widehat{e^{i0}}$ symbiotic unit circle $1^{\circlearrowleft}_{\odot} \in \mathbb{C}^1_1$, radius magnitude $|1^{\circlearrowleft}_{\odot}| = e^0 = 1$, $1 \in \mathbb{C}$.
This is an example of one complex basis vector $1^{\circlearrowleft}_{\odot} \in \mathbb{C}^1_1$, forming the linear vector space ***quality*** of complex number plane as a linear form over the complex scalars $\mathbb{C}$

(4.30) $$z = \psi 1^{\circlearrowleft}_{\odot} \quad \in \mathbb{C}^1_1, \quad \text{where } z \in \mathbb{C}^1_1 \quad \text{and the scalars } \psi \in \mathbb{C}, \quad \mathbb{C}^1_1 = \operatorname{span}\{1^{\circlearrowleft}_{\odot}\}, \quad \dim(\mathbb{C}^1_1) = 1.$$

We compare this complex linear form from a basis (4.30) with the Cartesian writing (4.27)

(4.31) $$\begin{array}{llll} a = x + y = \lambda_x \hat{1}_x + \lambda_y \hat{1}_y \in \mathbb{R}^2_{xy}, & \lambda_x, \lambda_y \in \mathbb{R} & \hat{1}_x, \hat{1}_y \in \mathbb{R}^2_{xy}, & \hat{1}_x \in \mathbb{R}^1_x,\ \hat{1}_y \in \mathbb{R}^1_y \\ \updownarrow \qquad\quad \updownarrow & \updownarrow & \updownarrow & \\ z = \psi 1^{\circlearrowleft}_{\odot} = \lambda_{\hat{1}_{\mathbb{C}}} \hat{1}_{\mathbb{C}} + \lambda_{\hat{\imath}_{\mathbb{C}}} \hat{\imath}_{\mathbb{C}} \in \mathbb{C}^1_1, \quad \psi \in \mathbb{C}, & \lambda_{\hat{1}_{\mathbb{C}}}, \lambda_{\hat{\imath}_{\mathbb{C}}} \in \mathbb{R} & \hat{1}_{\mathbb{C}}, \hat{\imath}_{\mathbb{C}} \in \mathbb{C}^1_1 & \sim \mathbb{R}^2_{xy} = \mathbb{R}^1_x \oplus \mathbb{R}^1_y \end{array}$$

The complex scalars $\mathbb{C}$ are divided into components we call the real and imaginary parts

(4.32) $$\operatorname{Re}(\psi) = \lambda_{\hat{1}} \in \mathbb{R}_x = \operatorname{Re}(\mathbb{C}), \qquad \operatorname{Im}(\psi) = \lambda_{\hat{\imath}} \in \mathbb{R}_y = \operatorname{Im}(\mathbb{C}), \qquad \text{and writes}$$

(4.33) $$z = \psi 1^{\circlearrowleft}_{\odot} = \operatorname{Re}(\psi) \hat{1}_{\mathbb{C}} + \operatorname{Im}(\psi) \hat{\imath}_{\mathbb{C}} \quad \in \mathbb{C}^1_1$$

We define for our intuition the complex plane by a basis set $\{1^{\circlearrowleft}_{\odot}\} \leftrightarrow \{\hat{1}_{\mathbb{C}}, \hat{\imath}_{\mathbb{C}}\} \leftrightarrow \{\hat{1}_x, \hat{1}_y\}$ from the implicit nature of the Cartesian plane by our traditional intuition of a xy-plane. This heuristic assumption will be elaborated to a much richer plane geometric algebra later in chapter 5.
An alternative description of the complex basis $\{1^{\circlearrowleft}_{\odot}\}$ is the polar expression $\psi = \rho e^{i\varphi} \in \mathbb{C}$, where $\rho, \varphi \in \mathbb{R}$, are already treated in the chapter on the concept of time[170] and will be explored further later below. It is worth noting that the linear forms

(4.34) $$z = \psi 1^{\circlearrowleft}_{\odot} = \rho e^{i\varphi} 1^{\circlearrowleft}_{\odot} \quad \in \mathbb{C}^1_1 \quad \text{for } \psi \in \mathbb{C} \quad \text{and} \quad \rho, \varphi \in \mathbb{R}$$

do not produce a geometric line, but rather a plane through the dilation of a circle $\rho = |\psi|$, and a rotation $e^{i\varphi} \in \mathbb{C}$ through the circle, when we have given $\psi = \rho e^{i\varphi}$. For the complex basis vector, we have $|1^{\circlearrowleft}_{\odot}| = |e^{i\varphi}| = 1$, with the speciality $|e^{i0}| = e^{i0} = 1 \in \mathbb{R}$
I also point out and claim, that although there is an isometry between the complex vector space and the real Cartesian plane $\mathbb{C}^1_1 \leftrightarrow \mathbb{R}^2_{xy}$, the substance (the system) of the complex vector space $\mathbb{C}^1_1$ containing more direct information about the structure of the plane than a Cartesian coordinate system of vector spaces $\mathbb{R}^2_{xy} = \mathbb{R}^1_x \oplus \mathbb{R}^1_y$. Some of the structure we hide in the term $1^{\circlearrowleft}_{\odot} \sim \hat{\imath}_{\mathbb{C}} \perp \hat{1}_{\mathbb{C}}$,
The (x,y) are full symmetric homogeneous coordinates, while the polar coordinates (ρ, φ) are inhomogeneous coordinates, where φ contain information from the rotation of the plane.
This problem will be developed later below through the Geometric Algebra.
The reader should consider the problem; what I mean by the heuristic definition
$\{1^{\circlearrowleft}_{\odot}\} \coloneqq \{e^{i\varphi}\} \sim \{\hat{\imath}_{\mathbb{C}} \perp \hat{1}_{\mathbb{C}}\} \sim \{\hat{1}_{\mathbb{C}}\}$ as the basis for the complex vector space

[170] Polar Coordinates (ρ, ϕ) referred to in sections 3.2.4.2 and later below in 5.3.2.2 and further.

For quotation reference use: ISBN-13: 978-8797246931