4.1. The Linear Algebraic Space

4.1.1. The Abstract Linear Space, a Vector Space

Regardless of the geometric perception of the natural space in physics, we define mathematics from the concept of numbers for an abstract set of mathematic objects that make up the elements of what we called a vector space V. The concept of numbers shall meet the requirements for an algebraic scalar field $\mathbb{K}$,[159] e.g., the real numbers $\mathbb{R}$.[160]
The elements of the algebraic scalars are designated with the Greek letters $\alpha, \beta, \lambda \in \mathbb{K}$.
The elements of the space V are called **linear vectors** if they full fill a so-called **linear additive algebra**. These elements are here in the abstract mathematical case referred to as lowercase Latin letters $a, b, c \in V$.

4.1.1.1. Algebra of a Linear Spaces

Generally, we define an arbitrary linear space $(V,\mathbb{K})\sim V$ often called a vector space V over scalars[161] $\mathbb{K}$. The linear space V has an additive neutral element, the *zero-vector* $0 \in V$.
For arbitrary elements $a, b, c \in V$, apply the following rules:

(4.1) $a+(b+c)=(a+b)+c$, the associative law for addition.

(4.2) $a+b=b+a$, the commutative law for addition.

(4.3) $\exists 0 \in V\colon\ a+0=a$ for $\forall a \in V$, the identical element of addition is the zero-vector 0.

(4.4) $\forall a \in V, \exists{-a} \in V \Rightarrow a+(-a)=0$, where $-a$ is the additive inverse of a.

(4.5) $\alpha(\beta b) = (\alpha\beta)b$ for $\alpha,\beta \in \mathbb{K}$, associative scalar multiplication, where $\mathbb{K}=\mathbb{Q},\mathbb{R},\mathbb{C}$.

(4.6) $1a=a$, where $1 \in \mathbb{K}$, identity by multiplying by a neutral scalar 1.

(4.7) $\lambda(a+b) = \lambda a+\lambda b$, $\lambda \in \mathbb{K}$, distributive scalar multiplication of vector addition.

(4.8) $(\alpha+\beta)c = \alpha c+\beta c$, for $\alpha,\beta \in \mathbb{K}$, distributive scalar multiplication of scalar addition.

(4.9) $\lambda a = a\lambda$, $\lambda \in \mathbb{K}$, commutative multiplication by scalars. – And extra:

(4.10) Subtraction of vectors deducted as $a-b=a+(-b)$

(4.11) Division of a vector with a scalar deducted as $\frac{a}{\alpha}=\frac{1}{\alpha}a$ from scalar division.

A set V (a ***quality***) that meets the additive algebraic conditions (4.1)-(4.11) – we call a **linear space** and this algebra a **linear algebra**. Such a linear space is often called a *vector space*, e.g., designated V. – These abstract *linear vectors* should not directly be confused with what we later below call for geometric vectors and Euclidean vectors, although these also perform the linear algebra (4.1)-(4.11).

4.1.1.2. The Dimensions of a Linear Algebra of a Linear Vector Space

Indexing with the natural numbers of vector elements: $a_1, a_2, \dots, a_i, \dots \in V$,
and the corresponding scalars: $\alpha_1, \alpha_2, \dots, \alpha_i, \dots \in \mathbb{K}$.
A linear space has dimension $n \in \mathbb{N}$ if there exists a set $a_1 \dots a_n$ of just n different prober vectors $a_i \neq 0$, for $i=1\dots n$, that are linearly independent, i.e.

(4.12) $\lambda_1 a_1+\cdots+\lambda_i a_i+\cdots+\lambda_n a_n=0 \Rightarrow \lambda_1=\cdots=\lambda_i=\cdots\lambda_n=0$, hence $\forall\lambda_i=0$.

A maximal set $a_1 \dots a_n$ of linearly independent vectors then form the basis for V_n.

[159] The original German name for an algebraic scalar field is *Zahl-Körper*, hence the name $\mathbb{K}$. To avoid misunderstandings when physicists use the word scalars in the plural the meaning is only one scalar field $\forall\alpha \in \mathbb{K}$. Therefore an algebraic scalar field consists only of one *number body* (*Zahl-Körper*). – If we use several ***qualitatively*** different dimensions of ***quantities***, maybe we will use different scalar fields to manage these ***qualities***.
[160] We considered the real numbers $\mathbb{R}$ known to the reader. Then the scalars $\mathbb{K}=\mathbb{R}$ is well known as a mathematical field concept.
[161] $\mathbb{K}$ is often omitted in $(V,\mathbb{K})$ and V is the term for a vector space over an implicit scalar field $\mathbb{K}$, which is given by the context.

We then say the vector space V_n has n linearly independent dimensions. We write

(4.13) $\dim(V_n)=n$, or $\dim(V_n,\mathbb{K}) = \dim_{\mathbb{K}}(V_n)=n$

Each vector $a \in V_n$ we will write as a linear combination of the basis set $a_1 \dots a_n$, as

(4.14) $a=\alpha_1 a_1+\cdots+\alpha_i a_i+\cdots+\alpha_n a_n$, where all $\alpha_i \in \mathbb{K}$.

The scalars $\alpha_i \in \mathbb{K}$ are called the coordinates of the vector a from the basis set $\{a_1 \dots a_n\} \subset V_n$.
This formula (4.14) produces what we call a linear span of the space $V_n=\operatorname{span}\{a_1 \dots a_n\}$, therefore, the term a linear space.[162]
Two proper vectors a and e in V_n are linearly dependent if there exists a scalar $\alpha \in \mathbb{K}$, so that $a=\alpha e$, that is $a+\lambda e=0 \Rightarrow \alpha=-\lambda \neq 0$. All vectors $a \in V_n$ which we can write in the form $a=\lambda e$ are said to belong to exactly one dimension U_1 in V_n, where $U_1 \subseteq V_n$.
U_1 we call a 1-dimensional subspace in V_n. For any $k \in \mathbb{N}$, where $0< k \leq n$, there exist a linearly independent set of vectors $a_{i_1} \dots a_{i_k}$, where $0<i_1<\dots<i_j<\dots<i_k\leq n$, for $j=1\dots k$, specially $a_0=0$. Then linear combination $u=\nu_1 a_{i_1}+\nu_2 a_{i_2}+\dots+\nu_k a_{i_k}$, for $\forall\nu_j \in \mathbb{K}$ belong to a subspace U_k of the linear space V_n, so that $u \in U_k \subseteq V_n$.
We say the subset $a_{i_1},\dots a_{i_k}$ spans the subspace U_k and writes $U_k=\operatorname{span}\{a_{i_1},\dots a_{i_k}\}$.
We noted that $(V_n,\mathbb{K})$ always has $U_0=\{0\}$ and V_n as subspaces.
For a proper subspace, we have $U_k \subset V_n$, i.e., $0< k <n$.

4.1.1.3. Sum of Subspaces

Given two linearly independent spaces U_r and U_s, we can form a direct sum $U_r \oplus U_s = V_n$, when $r+s=n$. U_r and U_s are then subspaces of V_n.
All basis set $a_1,\dots a_r$ for U_r and $a_{r+1},\dots a_{r+s}$ for U_s shall apply to (4.12) in accordance with the total basis set $a_1,\dots a_r,\dots a_n$ for V_n.
Do we have n one-dimensional subspace $X_{1_i} \subseteq V_n$ each with indices $i=1\dots n$, we have $\underbrace{X_{1_1}\oplus\dots X_{1_i}\oplus\dots X_{1_n}}_{n}=V_n$, equivalent to the linear form (4.14) where $a \in V_n$, when $a_i \in X_{1_i}$.
We see that each dimension has its own 1-dimensional space, which altogether constitute a n-dimensional space. We say the linear form generate the linear space V_n from the basis set $\{a_1,\dots,a_n\}$ or span by the designation $V_n=\operatorname{span}\{a_1,\dots,a_n\}$.

4.1.1.4. The Simplest Linear Space for a *Quality* of Physics

We look at the general linear space $(V,\mathbb{K})\sim V$. The simplest version of a linear space is the space over the scalars $\mathbb{K}$ themselves when we write $\mathbb{K}^1\sim(\mathbb{K}^1,\mathbb{K})$.
Here we will distinguish between the total subject of the scalars by the term $\mathbb{K}$ and the object for our intuition of the linear vector space $\mathbb{K}^1$.
The natural basis for $\mathbb{K}$ is the multiplicative neutral scalar $1 \in \mathbb{K}$.
$\mathbb{K}$ has then the natural basis set $\{\hat{1}\}$, and linear form $\lambda=\lambda 1$ produces its own abstract linear space $\mathbb{K}$ as a scalar substance. The individual scalars $\lambda \in \mathbb{K}$ as their total set $\mathbb{K}$ has no form for extension in the natural world. From this I declare – a controversial ethic[163]
– a priori synthetic judgment: $\mathbb{K}$ has no epistemological intelligible recognisable dimension,

(4.15) $Dim(\mathbb{K})=0$, the pure scalars $\mathbb{K}$ has no vector dimension in our intuition!

Can we in contrast construct an objective basis vector that is the recognisable for our intuition, for the vector space $\mathbb{K}^1$, called $\hat{1} \in \mathbb{K}^1$, then we have a basis set $\{\hat{1}\}$ for $(\mathbb{K}^1,\mathbb{K})\sim\mathbb{K}^1$.
The 1-dimensional linear form for objective space $\mathbb{K}^1$ is then

(4.16) $a=\lambda\hat{1}$, where $a \in \mathbb{K}^1$, and $\lambda \in \mathbb{K}$, from the generator $\hat{1} \in \mathbb{K}^1$, with $\dim_{\mathbb{K}}(\mathbb{K}^1)=1$.

[162] The recommendation is to abstract from the intuition of n straight line as a coordinate axis, and just stick to the linear form (4.14) as a span of V_n from an abstract basis $\{a_1 \dots a_n\} \subset V_n$, without any intuition to ***qualities*** in a natural world. (Pure mathematics.)
[163] This controversial ethical assumption appears more clearly when all the geometric algebra is described later below.

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