(3.101)
$$\hat{N}_+ := a_+^\dagger a_+ ,\qquad \hat{N}_- := a_-^\dagger a_- .$$

Afterwards, instead of (3.82)-(3.84) we write the Hamilton eigenvalue equation as

(3.102)
$$\hat{H}_\omega|n_+n_-\rangle = \hbar\omega\left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right)|n_+n_-\rangle = \hbar\omega\left(\hat{N}_+ + \hat{N}_- + 1\right)|n_+n_-\rangle \doteq \hbar\omega(n_+ + n_- + 1)|n_+n_-\rangle.$$

Here, it is noted that, like (3.36), that $n_+ \geq 0$ and $n_- \geq 0$.
In addition, we can rewrite the angular momentum (3.88) with the number operator of the rotation

(3.103)
$$\hat{L}_3 = -i\hbar\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) = \left(a_+^\dagger a_+ - a_-^\dagger a_-\right)\hbar .$$

This reformulates the eigenvalue equation for the angular momentum

(3.104)
$$\hat{L}_3|n_+n_-\rangle = \hbar\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)|n_+n_-\rangle = \hbar\left(\hat{N}_+ - \hat{N}_-\right)|n_+n_-\rangle \doteq \hbar(n_+ - n_-)|n_+n_-\rangle$$

The angular momentum operator $\hat{L}_3$ and the Hamilton operator $\hat{H}_\omega$ has by (3.89) the same eigenstates $|n_+n_-\rangle$ of the rotor oscillator. When the rotating circular oscillator description indices $(+,-)$ represent the same physical conditions as the Cartesian two-dimensional indices (1,2) for the harmonic oscillator (3.82) that had been the tradition in the 20'th century quantum mechanics, now here we must assume that the states are the same

(3.105)
$$|n_+n_-\rangle \sim |n_1n_2\rangle.$$

This despite the fact we know[84] that $|n_+\rangle \nsim |n_1\rangle$, $|n_-\rangle \nsim |n_2\rangle$, $|n_+\rangle \nsim |n_2\rangle$, $|n_-\rangle \nsim |n_1\rangle$.
For the ground state, we note

(3.106)
$$|0,0\rangle = |0_+0_-\rangle = |0_10_2\rangle$$

Then we can write the excited states in accordance with (3.41) as

(3.107)
$$|n_+n_-\rangle = \sqrt{\frac{1}{n_+!}}\sqrt{\frac{1}{n_-!}}\left(a_+^\dagger\right)^{n_+}\left(a_-^\dagger\right)^{n_-}|0,0\rangle$$

3.2.4. The Circular Rotating Oscillator Eigenvalues

Now we choose to describe the rotating plane oscillator with two new quantum numbers

(3.108)
$$n = n_+ + n_- \geq 0,\quad n \in \mathbb{N} \qquad \text{and} \qquad m = n_+ - n_- \in [-n, n] \subset \mathbb{N}$$

By this together with (3.102) and (3.104), the **Hamilton** and the **angular momentum** eigenvalue equations can be written

(3.109)
$$\hat{H}_\omega|n_+n_-\rangle \doteq \hbar\omega(n+1)|n_+n_-\rangle$$

(3.110)
$$\hat{L}_3|n_+n_-\rangle \doteq \hbar m\,|n_+n_-\rangle$$

n and m are independent and specify together a unique eigenstate. For a given n and m, we see that if n_+ increased by one, then correspondingly n_- decreased by one.
The ground state applies $n{=}0 \Rightarrow m{=}0$.
The first excited energy eigenstate applies $n = 1 \Rightarrow m = \pm 1$,
which is twice degenerated with the angular momentum either $+\hbar 1$ or $-\hbar 1$.
In this way for higher given n, the quantum number m jumps with 2

(3.111)
$$m = \quad -n,\quad -n+2,\quad -n+4,\quad \ldots\quad n-4,\quad n-2,\quad n$$

We see that energy eigenstates are degenerated with $n+1$ different values of m.
The name of eigenstate can be reformulated as follows

(3.112)
$$|n, m\rangle \sim |n_+n_-\rangle = \left|n_+{=}\frac{n+m}{2},\ n_-{=}\frac{n-m}{2}\right\rangle.$$

[84] The reason is that $|n_\pm,0\rangle$ is plane circular, and $|n_j\rangle$ is geometric linear in its perpendicular Cartesian approach.

The eigenvalue equations of the rotation plane oscillator for the **Hamilton operator** is written as

(3.113)
$$\hat{H}_\omega|n, m\rangle \doteq \hbar\omega(n+1)|n, m\rangle, \qquad \text{where} \qquad n = n_+ + n_- ,$$

and for the angular momentum operator

(3.114)
$$\hat{L}_3|n, m\rangle \doteq \hbar m\,|n, m\rangle, \qquad \text{where} \qquad m = n_+ - n_-$$

We repeat from (3.102) and (3.103), that the operators of this are
$\hat{H}_\omega = \hbar\omega\left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right)$ and $\hat{L}_3 = \hbar\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$, and compare with the stationary Schrödinger eigenvalue equation (2.67) $\hat{H}_\omega|\psi(t)\rangle \doteq E_\omega\,|\psi(t)\rangle$ and rewrite (3.82) as $\hat{H}_\omega\,|n, m\rangle \doteq E_{\omega,n}|n, m\rangle$, which leads to the energy eigenvalues

(3.115)
$$E_{\omega,n} = (n+1)\hbar\omega,$$

We look at the ground state energy $E_{\omega,0} = \hbar\omega$. We assume that the ground state is without action - nothing happens. Therefore, I claim the synthetic judgment:
The idea of an angular frequency ω of a rotation circle oscillation in the plane is a noumenon (as a platonic concept) for any virtual energy $E_{\omega,0}{=}\ \hbar\omega$ of the potential rotation plane with an expected angular frequency ω. – Anyway, I prefer $\hbar{=}1$ in such an abstraction concept.

3.2.4.2. The Rotating Circle Oscillator in Polar Coordinates

When we look at the angular momentum, we can form a more natural coordinate representation for a circle oscillator by polar ***quantities*** $(\rho, \phi) \in \mathbb{R}^2_{\circlearrowleft}$, rather than the Cartesian ***quantities*** $(q_1, q_2) \in \mathbb{R}^2_\perp$. From formula (3.48)

(3.116)
$$q^*_\omega(r, t) = re^{i\omega t} = r\cos(\omega t) + i\,r\sin(\omega t) \in \mathbb{C}$$

with the two Cartesian dimensions (3.49) $q_1{=}r\cos(\phi)$ and $q_2{=}r\sin(\phi)$, where $\phi{=}\omega t$ in which, we have the classic image, wherein a circle oscillation the amplitude-radius $r = \sqrt{q_1^2 + q_2^2} \geq 0$ is assumed with constant mean average $\overline{r} = \langle\tilde{r}(\rho)\rangle$, and $\frac{\partial\tilde{r}}{\partial\phi} = \frac{\partial\tilde{r}}{\partial t} = 0$.
The new intuition is that, the real magnitude ***quantity*** $\tilde{r}(\rho) \in \mathbb{R}$ designed as a distribution function over a stochastic real radial polar coordinate $\forall\rho \in \mathbb{R}$ for the circular rotation oscillator Ψ_ω

(3.117)
$$q^*_\omega(\rho, t) = \tilde{r}(\rho)\,e^{i\omega t} \in \mathbb{C},$$

where the angular ***quantity*** $\phi{=}\omega t \in \mathbb{R}$ is the angle in the plane for the center polar coordinates

(3.118)
$$q^*(\rho, \phi) = \tilde{r}(\rho)\,e^{i\phi} = \tilde{r}(-\rho)\,e^{i(\phi+\pi)} = \tilde{r}(-\rho)e^{i\phi}e^{i\pi} = -\tilde{r}(-\rho)e^{i\phi}, \quad \text{as } e^{i\pi} = -1 .$$

Here, it is noted, that a half-circle of rotation shifts the sign of the radial coordinate $(\rho, \phi) \leftrightarrow (-\rho, \phi + \pi + 2\pi n)$, a so-called ***parity inversion***,[85] of the radial coordinate $\rho \leftrightarrow -\rho$ **in a plane,** and the angular coordinate is periodic modulo 2π, $(\rho, \phi) \leftrightarrow (\rho, \phi + 2\pi n)$.

(3.119)
$$q^*(\rho, \phi) = q^*(\rho, \phi + 2\pi n) = \tilde{r}(\rho)\,e^{i\phi} = \tilde{r}(\rho)\,e^{i(\phi+2\pi n)}, \quad \text{for } \forall n \in \mathbb{Z}, \text{ periodical conserved.}$$

Usually, the definition interval is limited for polar coordinates to $\{\rho \in [0, \infty[,\ \phi \in [0, 2\pi[\,\}$, but here we will allow redundancy and thus allow all real coordinates $(\rho \in \mathbb{R},\ \phi \in \mathbb{R})$.
We not only allow $e^{i\omega t}$ just to repeat itself through the parameter t, but also that the radial *parity operation* $e^{i\pi} = -1$ as an *inversion in the plane* must be antisymmetric anti-identical

(3.120)
$$\tilde{r}(\rho) = -\tilde{r}(-\rho) \quad \in \mathbb{R},$$

through a center. This means, the real radial function as the object in a plane for an oscillator ***entity*** Ψ_ω must be an *odd function*. Here it is noted that the two polar radial parity antagonists balance each other $\tilde{r}(\rho) + \tilde{r}(-\rho) = 0$ in accordance with Newton's third law,[86] while the anti-symmetric difference represents the internal plane stress field probability distribution

[85] To be precise we will below in this book call a ***parity inversion*** operation for an *extension parity inversion of* ***first grade directions***.
[86] Newton's third law is following a line parity operation as inverse balance along a straight line.