3.2. The Two-Dimensional Quantum Harmonic Oscillator

3.2.1. The Plane Super-positioned Hamilton Operator

Now we are starting from the Hamilton operator for the harmonic oscillator in one dimension (3.5) and writes the circular harmonic oscillator in two orthogonal line dimensions

(3.80) $$\hat{H}_\omega = \hat{H}_{\omega,1} + \hat{H}_{\omega,2} = \frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial q_1^2} + q_1^2\right) + \frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial q_2^2} + q_2^2\right).$$

This we rewrite with annihilation- and creation-operators from (3.16) and further the number operator (3.30) and (3.33) for the two-dimensional Hamilton operator

(3.81) $$\hat{H}_\omega = \hbar\omega\left(a_1^\dagger a_1 + \tfrac{1}{2}\right) + \hbar\omega\left(a_2^\dagger a_2 + \tfrac{1}{2}\right) = \left(\hat{N}_1 + \hat{N}_2 + 1\right)\hbar\omega$$

Using the eigenvalue equation (3.21) rewritten with the notation $|n\rangle := |\psi_n\rangle$,

$\hat{H}_{\omega,i}|n_i\rangle \doteq E_{\omega,n_i}|n_i\rangle$, with eigenvalues from (3.29) $E_{\omega,n_i} = \left(n_i + \frac{1}{2}\right)\hbar\omega$

and further with a composite state $|n_1, n_2\rangle := |n_1\rangle|n_2\rangle = |n_1\rangle \otimes |n_2\rangle$, in the double Hilbert space $\mathcal{H}\otimes\mathcal{H}$ (§ 2.2.3, 2.2.4) that also is a Hilbert space, we can write the eigenvalue equation

(3.82) $$\hat{H}_\omega|n_1, n_2\rangle \doteq E_{\omega,n_1,n_2}|n_1, n_2\rangle$$

with the following energy eigenvalues for the excited states

(3.83) $$E_{\omega,n_1,n_2} = (n_1 + n_2 + 1)\hbar\omega$$

The excited states $|n\rangle = |n_1, n_2\rangle$ for $n = n_1 + n_2$ of the two-dimensional plane harmonic circle oscillator have eigenvalues

(3.84) $$E_{\omega,n} = (n+1)\hbar\omega$$

The ground state n_1=0 and n_2=0 $\Rightarrow$ n=0 provides just eigenvalues $E_{\omega,0} = \hbar\omega \in \mathbb{R}$.
The next state $E_{\omega,1} = \hbar\omega + \hbar\omega$ has degeneration in two cases

(3.85) $$(n_1{=}1 \wedge n_2{=}0) \text{ and } (n_1{=}0 \wedge n_2{=}1).$$

The degeneration is $(n+1)$ of the state $|n\rangle = |n_1, n_2\rangle$ is illustrated by the following table

Circular			$n = n_1 + n_2$	0	1	1	2	2	2	3	3	3	3	4	4	4	4	4	…
↔	↕	↺	n_1	0	1	0	1	2	0	3	0	2	1	4	0	3	1	2	…
↕	↔	↻	n_2	0	0	1	1	0	2	0	3	1	2	0	4	1	3	2	…
Degeneration			$n+1$	1	2		3			4				5					…

as, $n_1 \geq 0$ and $n_2 \geq 0$ by (3.36) we have $n \geq 0$.

3.2.2. The Angular Momentum Operator

From the classic angular momentum written as a vector product $\vec{L} = \vec{r} \times \vec{p}$, we write the angular momentum coordinate along the rotational axis using coordinates of the circle rotation in its transversal plane in accordance with the first expression in (3.66)

(3.86) $$L_3 = (q_1 p_2 - q_2 p_1)$$

The ***quantities*** q and p can be rewritten from (3.2)-(3.3) using the formulation (3.13) as operators

(3.87) $$\hat{q}_j = \frac{1}{\sqrt{2}}\left(a_j + a_j{}^\dagger\right) \sim q_j \quad \text{and} \quad \hat{p}_j = i\frac{1}{\sqrt{2}}\left(a_j{}^\dagger - a_j\right) \sim \frac{1}{\hbar}p_j.$$

By this, we write the ***quantum*** **angular momentum operator**

(3.88) $$\hat{L}_3 = i\frac{\hbar}{2}\left(a_1 + a_1^\dagger\right)\left(a_2^\dagger - a_2\right) - i\frac{\hbar}{2}\left(a_2 + a_2^\dagger\right)\left(a_1^\dagger - a_1\right) = -i\,\hbar\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)$$

Here, we have used the commutator relations $\left[a_1^\dagger, a_2^\dagger\right] = [a_2, a_1] = \left[a_1^\dagger, a_2\right] = \left[a_2^\dagger, a_1\right] = 0$, see (2.71).

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As we notice, that we also apply $\left[a_1^\dagger a_2, a_1^\dagger a_1 + a_2^\dagger a_2\right] = 0$ and $\left[a_2^\dagger a_1, a_1^\dagger a_1 + a_2^\dagger a_2\right] = 0$, we can deduce that $\hat{L}_3$ commute with Hamilton operator $\hat{H}_\omega$ (3.81)

(3.89) $$\left[\hat{L}_3, \hat{H}_\omega\right] = 0$$

Thereby eigenstates for $\hat{H}_\omega$, will also be eigenstates for $\hat{L}_3$, we just need to find a common reference basis in a double Hilbert space $\mathcal{H}\otimes\mathcal{H}$, consisting of stationary states.
Energy eigenvalues $E_{\omega,n} = (n+1)\hbar\omega$ above (3.84) represent degenerated states- modes (3.82), and it will be natural to let the degeneration correspond to individual eigenvalues of the angular momentum operator $\hat{L}_3$.
Above we took a start from the two field coordinate dimensions represented by (q_1, q_2) with the corresponding quantum operators a_1, a_2, $a_1^\dagger$, $a_2^\dagger$.
We have two degenerated oscillating rotation options:
The progressive oscillator rotation we wrote in (3.48) as the conjugate from (1.60)

(3.90) $$q_\omega^*(t) = re^{i\omega t} = r\cos(\omega t) + i\,r\sin(\omega t) \quad \in \mathbb{C}.$$

That is supplemented by the retrograde oscillator rotation, which then becomes

(3.91) $$q_\omega(t) = re^{-i\omega t} = r\cos(\omega t) - i\,r\sin(\omega t) \quad \in \mathbb{C}$$

From the above-mentioned degeneration of states (3.85) $(n+1) = 2$ for n=1, we also see a two-degeneration associated with rotation, progressive and retrograde. Therefore, we will indices with the value $+$ or $-$ for orientation of rotation. For rotation ***quantities*** we write $q_{\omega,+} = q_\omega^*(t) = re^{i\omega t}$ respectively $q_{\omega,-} = q_\omega(t) = re^{-i\omega t}$.

3.2.3. Ladder Operators of the Plane Quantum Mechanical Harmonic Circle Oscillator

To find a new[83] formulation of ***quantised*** eigenstates for $\hat{L}_3$
we introduce the two annihilation operators for rotation.

(3.92) $$a_+ := \frac{1}{\sqrt{2}}(a_1 - ia_2) \ ,$$

(3.93) $$a_- := \frac{1}{\sqrt{2}}(a_1 + ia_2) \ .$$

and the corresponding two Hermitian adjoined creation operators

(3.94) $$a_+^\dagger := \frac{1}{\sqrt{2}}\left(a_1^\dagger + ia_2^\dagger\right) \ ,$$

(3.95) $$a_-^\dagger := \frac{1}{\sqrt{2}}\left(a_1^\dagger - ia_2^\dagger\right) \ .$$

For these we note the following commutator relations

(3.96) $$\left[a_+, a_+^\dagger\right] = \left[a_-, a_-^\dagger\right] = 1.$$

(3.97) $$\left[a_+^\dagger, a_-^\dagger\right] = [a_+, a_-] = \left[a_+^\dagger, a_-\right] = \left[a_+, a_-^\dagger\right] = 0.$$

From definitions (3.93) to (3.95), we get

(3.98) $$a_+^\dagger a_+ = \frac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2 + ia_2^\dagger a_1 - ia_1^\dagger a_2\right)$$

(3.99) $$a_-^\dagger a_- = \frac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2 - ia_2^\dagger a_1 + ia_1^\dagger a_2\right)$$

By adding these two equations we get the number operators, see also (3.30)

(3.100) $$a_+^\dagger a_+ + a_-^\dagger a_- = a_1^\dagger a_1 + a_2^\dagger a_2 \sim \hat{N}_+ + \hat{N}_- = \hat{N}_1 + \hat{N}_2$$

We have introduced two new number operators for the rotation circle oscillator

[83] This development is inspired by [8] ← taken from [7]

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