3.1.8. Classical Angular Momentum

From classical physics, we have that the angular momentum can be written (by Gips cross product)

(3.62) $\vec{L} = \vec{r} \times \vec{p}$.

By the coordinates of the circle oscillation in the transversal plane of the rotation axis, this is written as a coordinate along the ***direction*** $\vec{\vec{\omega}}$ of this rotation axis, the third axis:

(3.63) $L_3 = (q_1 p_2 - q_2 p_1)$,

where, the three Cartesian coordinates apply

(3.64) $\vec{q} \sim (q_1, q_2, 0) \in \mathbb{R}^3_\perp$, $\quad \vec{p} \sim (p_1, p_2, 0) \in \mathbb{R}^3_\perp$ $\quad$ and $\quad \vec{L} = \vec{L}_3 \sim (0,0,L_3) \in \mathbb{R}^3_\perp$.

Just as we in section 1.7.6 introduced the inertia factor oscillator $m_\omega \leftarrow I$ in defining the internal momentum $p_\omega = m_\omega \dot{q}_\omega = m_\omega \frac{\partial}{\partial t} q_\omega = -i\omega m_\omega q_\omega$, (1.76), we will look at a similar ratio of the angular momentum. We write

(3.65) $L_3 = I_3 \omega$,

where I_3 is a factor for the moment of inertia.

For an axial rotation, the particular moment of inertia[79] is characterised as $I_3 = mr^2$, where we count an inertia factor m to a certain radius r, we can write the angular momentum as

(3.66) $L_{3,r} = (q_1 p_2 - q_2 p_1) = I_{3,r}\omega = I_{3,r}(q_1\dot{q}_2 - q_2\dot{q}_1)/r^2 = mr^2(q_1\dot{q}_2 - q_2\dot{q}_1)/r^2 = m(q_1\dot{q}_2 - q_2\dot{q}_1)$.

Just as we for the Hamilton function avoided $\dot{q}$, we will continue with the momentum p, that implicitly hides m and thereby also hides I_3 in the expression for $L_3 = q_1 p_2 - q_2 p_1$.

3.1.9. Quantising the Angular Momentum

Here in section 3.1.7 above the regular circular motion is outlined in a classic image.

The cyclic circle oscillator in the transversal plane is described in two dimensions (3.49) for points $(q_{\omega,1}, q_{\omega,2}) \in \mathbb{R}^2_\perp$ and at (3.51) for points $(\dot{q}_{\omega,1}, \dot{q}_{\omega,2}) \in \mathbb{R}^2_\perp$, as we transform to the momentum points $(p_{\omega,1}, p_{\omega,2}) \in \mathbb{R}^2_\perp$ according to (3.66).

All these points are judged to be in the same transversal plane for the physical ***entity*** Ψ_ω .

As the classical ***quantities*** in (3.63)-(3.66) implicitly depend on ω in Ψ_ω, we here marked them with ω as indices just as introduced in (3.49).

Now we will ***quantise*** these classical field ***quantities*** by the transformation (3.2)

(3.67) $q_j \sim \sqrt{\frac{m\omega}{\hbar}} \cdot |q_{\omega,j}|$, $\quad$ and their canonical conjugated *mode* operators from (3.3)

(3.68) $\hat{p}_j = -i\frac{\partial}{\partial q_j}$, $\quad$ with $j = 1{,}2$

Inspired by [7], [8] this will now be further developed directly from the two real scalar fields q_1 and q_2 called the ***quantum mechanical real field quantities.***

In the plane for the abstract idea (noumenon) of the circle oscillator we are assuming the two Cartesian perpendicular and algebraic orthogonal real scalar field ***quantities*** q_1 and q_2 as isotropic, in the sense of isometric measure with the same standard $|q_{\omega,1}| = |q_{\omega,2}|$, so the idea of a circle[80] in the plane are met. We must maintain our intuition that all these objects (points) of types (q_1, q_2) and $(\hat{p}_1, \hat{p}_2)$ are belonging to the same transversal plane perpendicular to a thought (noumenon) rotation axis, that we consider orthogonal in our algebraic calculations. –

[79] For an axial rotating disc with the mass m and radius r the moment of inertia $I_3 = \frac{1}{2}mr^2$. In general, for a rigid body with a maximum radius R from CM, $I_3 = Cmr^2$, where $0<C\leq1$, which is dependent on the distribution of the mass.

[80] This means that ellipses or elliptical oscillations, etc. not allowed for this symmetry just for this circular oscillator idea.

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Edition 2, © 2020-22

3.1.9.2. Differential Operator in 3 Dimensions

For later use, we will associate these with a vector operator form $\vec{q}$ and $\vec{p} = -i\hbar\nabla$, where

(3.69) $\nabla = \sum_j \frac{\partial}{\partial q_j} = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3}$,

then the angular momentum vector operator will look like this[81]

(3.70) $\vec{L} = \vec{q} \times \vec{p} = \vec{q} \times (-i\hbar\nabla) = -i\hbar(\vec{q} \times \nabla)$

To find an angular momentum operator component we use e.g., (3.63)

(3.71) $L_3 = (q_1 p_2 - q_2 p_1) \leftrightarrow \quad \hat{L}_3 = (\hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1) = -i\hbar\left(\hat{q}_1\frac{\partial}{\partial q_2} - \hat{q}_2\frac{\partial}{\partial q_1}\right)$,

(3.72) $L_2 = (q_3 p_1 - q_1 p_3) \leftrightarrow \quad \hat{L}_2 = (\hat{q}_3\hat{p}_1 - \hat{q}_1\hat{p}_3) = -i\hbar\left(\hat{q}_3\frac{\partial}{\partial q_1} - \hat{q}_1\frac{\partial}{\partial q_3}\right)$,

(3.73) $L_1 = (q_2 p_3 - q_3 p_2) \leftrightarrow \quad \hat{L}_1 = (\hat{q}_2\hat{p}_3 - \hat{q}_3\hat{p}_2) = -i\hbar\left(\hat{q}_2\frac{\partial}{\partial q_3} - \hat{q}_3\frac{\partial}{\partial q_2}\right)$.

Using the commutator relation (2.71) $[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}$, $[\hat{q}_j, \hat{q}_k] = 0$, $[\hat{p}_j, \hat{p}_k] = 0$

we can find the commutator relations for these angular momentum components like [9](11.4)

(3.74) $[\hat{L}_1, \hat{q}_3] = [(\hat{q}_2\hat{p}_3 - \hat{q}_3\hat{p}_2), \hat{q}_3] = +\hat{q}_2[\hat{p}_3, \hat{q}_3] = -i\hbar\hat{q}_2$, $\quad [\hat{L}_1, \hat{q}_1] = 0$,

(3.75) $[\hat{L}_1, \hat{p}_3] = [(\hat{q}_2\hat{p}_3 - \hat{q}_3\hat{p}_2), \hat{p}_3] = -[\hat{q}_2, \hat{p}_3]\hat{p}_2 = -i\hbar\hat{p}_2$, $\quad [\hat{L}_1, \hat{p}_1] = 0$,

and from this the commutation of angular components

(3.76) $[\hat{L}_1, \hat{L}_2] = [\hat{L}_1, (\hat{q}_3\hat{p}_1 - \hat{q}_1\hat{p}_3)] = [\hat{L}_1, \hat{q}_3]\hat{p}_1 - \hat{q}_1[\hat{L}_1, \hat{p}_3] = -i\hbar\hat{q}_2\hat{p}_1 + i\hbar\hat{q}_1\hat{p}_2 = i\hbar\hat{L}_3$

and similar permutating the indices 1→2→3→1 we get

(3.77) $\boxed{[\hat{L}_1, \hat{L}_2] = i\hbar\hat{L}_3, \quad [\hat{L}_2, \hat{L}_3] = i\hbar\hat{L}_1, \quad [\hat{L}_3, \hat{L}_1] = i\hbar\hat{L}_2}$ $\quad$ as [9](11.5)

We see that the three different angular momentum components for one ***entity*** do not commute, even though we see they are strongly interconnected.

In the rest of this chapter 3, we will look at one ***direction*** from the past to the future. We will represent this by one angular momentum ***direction*** vector $\vec{L}_3$ that in principle is free in space. This leads us to presume, that the other transversal components $\hat{L}_1 \rightarrow \vec{L}_1 = 0$ and $\hat{L}_2 \rightarrow \vec{L}_2 = 0$ vanish. In this case, cyclic things happen in the transversal plane with indices 1↔2 and we concentrate our efforts around (3.66)→(3.71)

(3.78) $\hat{L}_3 = (\hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1) = -i\hbar\left(\hat{q}_1\frac{\partial}{\partial q_2} - \hat{q}_2\frac{\partial}{\partial q_1}\right)$

Therefore, we will interpret the fundamental cyclic angular mode as a unit circle oscillation in one plane. (In the central particular case Kepler conic section is aloud, special periodical elliptical cycles are required to achieve an oscillation).

As you may already know, Kepler's Second Law is the most fundamental law of physic:

- A line segment $\mathbf{q}$ joining a planet and the Sun sweeps out equal angular areas $\mathbf{q}\wedge\nabla$ per chronometric measure, –[82]

We will elaborate further on this fundament throughout this book.

- When we later in chapter 6.5 look into geometric algebra you may interpret an angular momentum operator (3.70) just as a bivector operator

(3.79) $\mathbf{L} = \hbar\mathbf{q} \wedge \nabla$.

That is a unit ***quantum*** if $|\mathbf{q}| = |\vec{q}| = 1$ and $\hbar = 1$. –

[81] For a better understanding of this, the reader can e.g., consult Merzbacher [9]C11,p.233ff. For later use see below (6.266)-(6.267).

[82] The product symbol ∧ is the Grassmann exterior (outer) product used later on in this book for ***qualities*** of higher grades, see ⤻. This is much simpler than the pseudo vector formulation (3.70) using the Gips cross product in a 'world' of pure 1-vectors.